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On the Analytical Solution of the S_N Radiative Transport Equation in a Slab for a Space-dependent Albedo Coefficient

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Abstract. In this work, we report a genuine general analytical solution for the linearized S_N radiative-conductive transfer problem in a heterogeneous plane parallel atmosphere with the albedo coefficient depending continuously on the spatial variable. By general solution, we mean that the solution is valid for an arbitrary albedo coefficient continuous functions of the spatial variable having the property of fulfill the requirements of existence and uniqueness. The key feature of this novel approach embodies the steps: following the idea of the Decomposition method, we transform the original problem into a set of recursive problems with constant albedo coefficients, having the main feature that the sources terms takes the information of the spatial dependency of the albedo coefficient into account. This procedure allows us to solve, analytically, the resulting recursive system by the LTS_N method developed for a constant albedo coefficient. Finally, we present the error control analysis of the solution and numerical comparisons against the literature results.

1. Introduction

Radiative transfer considers problems that model the physical phenomenon of energy transfer by radiation in media. These phenomenon appears in variety of realms including engineering applications like heat transport by radiation [1]. The nonlinearity of this equation comes from the local thermal description using the Stefan-Boltzmann law that is related to heat transport by radiation which in turn is related to the radiation intensity and renders the radiative transfer problem a radiative-conductive one [2, 3]. Solutions in the literature are basically linearized and of numerical nature [4, 5, 6, 7, 8, 9]. However, it is worth mentioning that a general solution from an analytical approach for this type of problems exists only in the discrete ordinate approximation and for homogeneous media as reported by Segatto et al. in reference [10]. Its basic idea comprehends the steps: following the idea of the Decomposition method [11, 12], the authors constructed a recursive system of linearized S_N radiative-conductive equations with the main feature that the nonlinearity information is carried out by the source term, unlike the well known Decomposition method [13] which expands the nonlinearity term in polynomials series ($\hat{A}_n$ and A_n polynomials). Further, the authors also splitted the operator associated to the equation considered, as a sum of the linear and non linear operators, but not likewise the Decomposition approach, which decomposes the original problem as the summation of the higher derivative, remaining linear and non linear operators. Morever the authors constructed a recursive system of differential, but not integral equations like the Decomposition technique, whith was then solved by the authors by the $\mathrm{LTS_N}$ method [10]. For more detais see the works of Segatto et al. [14], and Gonçalves et al. [15]. Motivated by the task of extending this of sort solution for heterogeneous media, in this work, we report a genuine general analytical solution for the linearized $\mathrm{S_N}$ radiative-conductive transfer problem in a heterogeneous plane parallel atmosphere with the albedo coefficient depending continuously on the spatial variable. By general solution, we mean that the solution is valid for an arbitrary albedo coefficient continuous functions of the spatial variable having the property of fulfill the requirements of existence and uniqueness [16, 17]. By analytical we mean that no approximation is done along the solution derivation. We are aware of the works of Ćengel and Öz̈işik [9], and Garcia and Siewert [18], but they are restricted for very specific albedo functions, namely polynomials and exponential functions. The key feature of this novel approach embodies the steps: procceeding in similar manner of the nonlinear problem discussed, we transform the original problem into a set of recursive problems with constant albedo coefficients, having the main feature that the sources terms takes the information of the spatial dependency of the albedo coefficient into account. This procedure allows us to solve, analytically, the resulting system by the $\mathrm{LTS_N}$ method developed for a constant albedo coefficient. Therefore, the series solution attained, allow us to obtain numerical results with a prescribed accuracy, by controlling the number of the terms in the series solution. To our best knowledge this sort of solution is not found in literature. We complete our analysis presenting simulations and comparisons against the literature results.

2. The Analytical Solution

To construct the analytical solution of the $\mathrm{S_N}$ radiative transfer in an inhomogeneous, emitting, absorbing and grey plane-parallel slab of optical thickness, with variable albedo coefficient with position, without source [19], let us consider the problem:

$$\mu\frac{\partial I}{\partial \tau}(\tau,\mu) + I(\tau,\mu) = \frac{\omega(\tau)}{2}\sum_{\ell=0}^{L}\beta_\ell P_\ell(\mu)\int_{-1}^{1} P_\ell(\mu')I(\tau,\mu')d\mu', \tag{1}$$

for $0 < \tau < \tau_0$ and to the ensuing reflecting and emitting boundary condition:

$$I(0,\tau) = f_1(\mu) + 2\rho_1\int_0^1 I(0,-\mu)\mu d\mu, \qquad \mu > 0 \tag{2}$$

and

$$I(\tau_0,\tau) = f_2(\mu) + 2\rho_2\int_0^1 I(\tau_0,\mu)\mu d\mu, \qquad \mu > 0. \tag{3}$$

Here $I(\tau,\mu)$ is radiation intensity, is the space-dependent albedo coefficient, n is refractive index of the medium, σ is the Stefan-Boltzmann constant, T_i, ϵ_i and ρ_i are respectively the temperature, the diffuse emissivity and reflectivity at the boundary. Moreover the expansions coefficients β_ℓ and the Legendre polynomials $P_\ell(\mu)$ and $P_\ell(\mu')$ are associated with the definition of the anisotropic scattering phase-function expressed as:

$$p(\mu,\mu') = \sum_{\ell=1}^{L}\beta_\ell(\mu)P_\ell(\mu)P_\ell(\mu') \tag{4}$$

here $a_0 = 1$. In order to determine the searched analytical solution, we now consider that the albedo coefficient has the form:

$$\omega(\tau) = \bar{\omega} + \omega_0(\tau) \tag{5}$$

From this assumption, we recast equation (1) like:

$$L(I) = \mu \frac{\partial I}{\partial \tau}(\tau,\mu) + I(\tau,\mu) - \frac{\bar{\omega}}{2} \sum_{\ell=0}^{L} \beta_\ell P_\ell(\mu') \int_{-1}^{1} P_\ell(\mu') I(\tau,\mu') d\mu' = S(I) \tag{6}$$

where the source $S(I)$ is written like:

$$S(I) = \frac{\omega_0(\tau)}{2} \sum_{\ell=0}^{L} \beta_\ell P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu) I(\tau,\mu) d\mu, \tag{7}$$

Following this idea of Decomposition method [11], we assume that the radiation intensity reads like:

$$I(\tau,\mu) = \sum_{m=1}^{M} I_m(\tau,\mu) \tag{8}$$

Replacing the expression (8) in equation (6) we come out with the following recursive system:

$$\begin{aligned} L(I_0) &= 0 \\ L(I_1) &= S(I_0) \\ &\vdots \\ L(I_m) &= S(I_{m-1}) \end{aligned} \tag{9}$$

for $0 \leq m \leq M$. At this point, we must remark that the first equation of the recursive system (9) satisfy the boundary condition given by equations (2) and (3), meanwhile the remaining equations fulfill the homogeneous boundary condition mentioned. From the previous discussion, we must realize that we reduce the solution of the original problem to a set of problems with constant albedo coefficient which are then readly solved, analytically, by the well known LTS$_\mathrm{N}$ method [14]. Here we need underline that the number of recursive problems to be solved (M) is governed by the prescribed accuracy to be reached. For sake of completeness, in sequel, we report the LTS$_\mathrm{N}$ solution for the set of recursive problems discussed:

$$\begin{aligned} \mathbf{I}(\tau) = \mathbf{X}\Big(\mathbf{E}^+(\mathbf{D}(\tau-\tau_0)) + \mathbf{E}^-(\mathbf{D}\tau)\Big)\mathbf{V} + \mathbf{X}\Big(\int_0^\tau (\mathbf{E}^-(\mathbf{D}(\tau-\xi))\mathbf{X}^{-1}\mathbf{S}(\xi)d\xi + \\ \int_{\tau_0}^{0} \mathbf{E}^+(\mathbf{D}(\tau-\xi))\mathbf{X}^{-1}\mathbf{S}(\xi)d\xi\Big)\mathbf{M} \end{aligned} \tag{10}$$

where $\mathbf{D}$ and $\mathbf{X}$ are respectively the diagonal eingenvalues matrix and the eigenvector matrix of the LTS$_\mathrm{N}$ matrix $\mathbf{A}$, whose entries are written as:

$$a(i,j) = \begin{cases} -\dfrac{1}{\mu_i} + \dfrac{\omega_j}{2\mu_i}\left[\displaystyle\sum_{\ell=0}^{L} \beta_\ell P_\ell(\mu_i) P_\ell(\mu_j)\right], & se\ i = j \\ \dfrac{\omega_j}{2\mu_i}\left[\displaystyle\sum_{\ell=0}^{L} \beta_\ell P_\ell(\mu_i) P_\ell(\mu_j)\right], & se\ i \neq j \end{cases} . \tag{11}$$

Further, from the Decomposition of the diagonal matrix $\mathbf{E}(\mathbf{D}(\tau))$:

$$\mathbf{E}(\mathbf{D}(\tau)) = \mathbf{E}^{+}(\mathbf{D}(\tau)) + \mathbf{E}^{-}(\mathbf{D}(\tau)) \tag{12}$$

we remark that the entries of $\mathbf{E}^{+}(\mathbf{D}(\tau))$ and $\mathbf{E}^{-}(\mathbf{D}(\tau))$ are respectively written like:

$$e_{ii}^{+}(\tau) = \begin{cases} \mathrm{e}^{d_i\tau}, & \text{if } d_i > 0 \\ 0, & \text{if } d_i < 0 \end{cases}, \qquad e_{i}^{-}(\tau) = \begin{cases} \mathrm{e}^{d_i\tau}, & \text{if } d_i < 0 \\ 0, & \text{if } d_i > 0 \end{cases}. \tag{13}$$

In addition the column vetor $\mathbf{M}$ has the form:

$$\mathbf{M} = \left[\frac{1}{\mu_1}\ \frac{1}{\mu_2}\ \ldots\ldots \frac{1}{\mu_N}\right]^T \tag{14}$$

Further the integration constant vector $\mathbf{V}$ is a vector which is determined by solving the linear system resulting form the application of the boundary condition. More information and details about this solution is found in the works of Segatto et al. [10] and Vilhena et al. [1].

3. Numerical Results

In order to make comparisons with available results in literature, in the sequel, without loosing generality, we specialize the application of the proposed methodology for the following problems:

In the first problem we focus our attention to show the convergence of the proposed solution for heterogeneous media ranging the number of equations in the recursive system, from 1 to 10. Bearing in mind the proved convergence of the $\mathrm{LTS_N}$ method when N goes to the infinitive[20], we report numerical simulations for $N = 400$. We will show in the next problem that this value for N (400) is suitable for our purpose. To this end, let us consider the following radiative transfer problem in a heterogeneous slab of thickness $L = 1$ mfp, assuming the albedo polynomial function, $\omega(x) = 1 - 1.4x + 0.6x^2$ and the boundary condition: $\rho_1 = \rho_2 = 0$, $f_1(\mu) = 1$, $f_2(\mu) = 0$. Looking for the results displayed in table 1, we promptly realize the numerical convergence of eight rounded places for the results attained for the heterogeneous problem considered. We reinforce this affirmative noticing that all the recursive problems are solved by the $\mathrm{LTS_{400}}$ approach.

Next we display in table 2, numerical comparisons of our results for the reflectivity (R) and transmissivity (T) for the first and second order polynomial albedo coefficient functions for the same problem against the one achieved by Cengel and Ozişik [9] . To highlight the generality of this method we also report in table 3 numerical comparisons against the ones attained by Garcia and Siewert [18].

Table 1. Numerical convergence of the recursive system solution for the radiation intensity Iteration in the discrete directions at $\tau = 0.5$ considering $\omega(x) = 1 - 1.4x + 0.6x^2$ from j varing fron 1 to 11

μ / j	1	2	3	4	5	6	7	8	9	10	11
-1.0	.0366944770	.0236823372	.0245079378	.0243117509	.0243222678	.0243192122	.0243193413	.0243192932	.0243192946	.0243192939	.0243192939
-0.9	.0398819082	.0258490716	.0267378842	.0265266208	.0265379305	.0265346402	.0265347790	.0265347272	.0265347288	.0265347279	.0265347279
-0.8	.0436610581	.0284477145	.0294092155	.0291806061	.0291928240	.0291892638	.0291894136	.0291893575	.0291893592	.0291893583	.0291893583
-0.7	.0482062356	.0316195548	.0326649665	.0324163223	.0324295832	.0324257112	.0324258736	.0324258127	.0324258145	.0324258135	.0324258135
-0.6	.0537618820	.0355734442	.0367156621	.0364438949	.0364583500	.0364541182	.0364542951	.0364542285	.0364542305	.0364542294	.0364542294
-0.5	.0606747269	.0406298726	.0418824412	.0415843215	.0416001217	.0415954802	.0415956733	.0415956002	.0415956023	.0415956012	.0415956012
-0.4	.0694319109	.0473017675	.0486747970	.0483479882	.0483652250	.0483601379	.0483603482	.0483602680	.0483602703	.0483602691	.0483602691
-0.3	.0806620851	.0564463479	.0579324707	.0575791624	.0575976758	.0575921786	.0575924037	.0575923171	.0575923196	.0575923182	.0575923182
-0.2	.0948542979	.0695411329	.0710686056	.0707083173	.0707270904	.0707214909	.0707217185	.0707216302	.0707216327	.0707216313	.0707216313
-0.1	.1108739241	.0891284580	.0904410684	.0901526219	.0901684948	.0901640374	.0901642331	.0901641631	.0901641653	.0901641642	.0901641642
-0.001	.1266859016	.1204017233	.1215924838	.1215444920	.1215624561	.1215619337	.1215622190	.1215622144	.1215622190	.1215622190	.1215622190
0.001	.1270398499	.1212319506	.1224408249	.1223998390	.1224182279	.1224178253	.1224181192	.1224181167	.1224181215	.1224181215	.1224181216
0.1	.1528223500	.1803601613	.1847253445	.1852558686	.1853351701	.1853449674	.1853463992	.1853465790	.1853466049	.1853466082	.1853466086
0.2	.2309256998	.2831752004	.2904668502	.2914434182	.2915770505	.2915948901	.2915973208	.2915976459	.2915976900	.2915976960	.2915976968
0.3	.3266824747	.3853596428	.3933408850	.3944271298	.3945734022	.3945931999	.3945958639	.3945962241	.3945962726	.3945962792	.3945962801
0.4	.4108631731	.4687840963	.4765796555	.4776465700	.4777893745	.4778088012	.4778114032	.4778117565	.4778118039	.4778118103	.4778118112
0.5	.4798176207	.5346918061	.5420349929	.5430426909	.5431771551	.5431954939	.5431979444	.5431982778	.5431983225	.5431983285	.5431983293
0.6	.5358043287	.5870787140	.5939154502	.5948550881	.5949802393	.5949973338	.5949996149	.5949999257	.5949999672	.5949999729	.5949999736
0.7	.5816238242	.6293573872	.6357062386	.6365796772	.6366958689	.6367117555	.6367138734	.6367141622	.6367142008	.6367142060	.6367142067
0.8	.6195873124	.6640431241	.6699453286	.6707578682	.6708658648	.6708806412	.6708826099	.6708828784	.6708829143	.6708829192	.6708829198
0.9	.6514483094	.6929403454	.6984414766	.6991991734	.6992998154	.6993135927	.6993154274	.6993156778	.6993157112	.6993157158	.6993157164
1.0	.6785135233	.7173490398	.7224923798	.7232010597	.7232951438	.7233080286	.7233097437	.7233099779	.7233100091	.7233100134	.7233100139

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Table 2. Numerical comparisons for reflectivity and transmisivity for the albedo polynomial albedo thickness $L = 1$ and transparent boundaries ($\rho_1 = \rho_2 = 0$, $f_1(\mu) = 1$, $f_2(\mu) = 0$)

	Cengel Results [9]		LTS_{200} Results		LTS_{300} Results		LTS_{400} Results	
$\omega(x)$	R	T	R	T	R	T	R	T
τ	0.064838	0.313115	0.064840	0.313119	0.064843	0.0313121	0.064845	0.313123
$0.1 + 0.8\tau$	0.076897	0.310771	0.076900	0.310775	0.076901	0.310776	0.076902	0.310777
$0.2 + 0.6\tau$	0.089796	0.308978	0.89801	0.308981	0.089800	0.308981	0.089801	0.308982
$0.3 + 0.4\tau$	0.103594	0.307712	0.103601	0.307715	0.103599	0.307714	0.103598	0.307715
$0.4 + 0.2\tau$	0118358	0.306959	0.1183367	0.306961	0.118362	0.306960	0.118361	0.306960
0.5	0.134165	0.306709	0.134175	0.306711	0.134170	0.306710	0.134168	0.306709
$0.6 - 0.2\tau$	0.151104	0.306959	0.151116	0.306960	0,151109	0.306959	0.151107	0.306959
$0.7 - 0.4\tau$	0.169279	0.307712	0.169292	0.307713	0.169284	0.307712	0.169281	0.307712
$0.8 - 0.6\tau$	0.188806	0.308978	0.188821	0.308979	0.188812	0.308977	0.188808	0.308977
$0.9 - 0.8\tau$	0.209824	0.310771	0.209841	0.310772	0.209831	0.310771	0.209827	0.310770
$1 - \tau$	0.232491	0.313115	0.232508	0.313115	0.232498	0.313115	0.232493	0.313114
$0.3 + 0.2\tau + 0.6\tau^2$	0.081072	0.311079	0.081078	0.311083	0.811078	0.311085	0.081078	0.311087
$0.4 - 0.2\tau + 0.6\tau^2$	0.108266	0.307972	0.108275	0.307975	0.108271	0.307975	0.108270	0.307976
$0.8 - \tau + 0.6\tau^2$	0.175240	0.307972	0.175255	0.307973	0.175246	0.0307972	0.175242	0.307971
$1 - 1.4\tau + 0.6\tau^2$	0.216635	0.311079	0.216652	0.311079	0.216642	0.311078	0.216637	0.311078

Table 3. The exit distribution $I(\tau, \mu)$ for $\omega(\tau) = exp(-\frac{\tau}{s})$

	$I(0, \mu)$							
μ	$s = 1$		$s = 10$		$s = 10^2$		$s = 10^3$	
	Results from [18]	LTS_{400}	Results from [18]	LTS_{400}	Results from [18]	LTS_{400}	Results from [18]	LTS_{400}
-1.0	.19055	.19055	.44517	.44517	.66146	.66146	.72904	.72904
-0.9	.20517	.20517	.46615	.46615	.67943	.67943	.74510	.74510
-0.8	.22223	.22223	.48910	.48910	.69805	.69805	.76139	.76139
-0.7	.24239	.24239	.51427	.51427	.71733	.71734	.77788	.77788
-0.6	.26656	.26656	.54197	.54197	.73730	.73730	.79455	.79455
-0.5	.29609	.29609	.57257	.57257	.75799	.75800	.81143	.81143
-0.4	.33296	.33296	.60647	.60647	.77946	.77946	.82857	.82857
-0.3	.38031	.38031	.64418	.64418	.80181	.80181	.84606	.84606
-0.2	.44328	.44328	.68632	.68632	.82523	.82523	.86410	.86410
-0.1	.53112	.53112	.73398	.73398	.85028	.85028	.88319	.88319
-0.05	.58966	.58966	.76081	.76082	.86400	.86400	.89361	.89361
	$I(5, \mu)$							
μ	$s = 1$		$s = 10$		$s = 10^2$		$s = 10^3$	
0.05	.6075 (-5)	.60745(-5)	.58031(-2)	.58026(-2)	.62883(-1)	.62881(-1)	.96845(-1)	.96843(-1)
0.1	.69252(-5)	.69250(-5)	.63702(-2)	.63700(-2)	.69024(-1)	.69023(-1)	.10629	.10629
0.2	.96423(-5)	.96422(-5)	.76183(-2)	.76181(-2)	.80567(-1)	.80567(-1)	.12363	.12363
0.3	.16234(-4)	.16234(-4)	.91482(-2)	.91481(-2)	.91918(-1)	.91917(-1)	.14012	.14012
0.4	.43858(-4)	.43858(-4)	.11119(-1)	.11119(-1)	.10342	.10342	.15622	.15622
0.5	.16937(-3)	.16937(-3)	.13725(-1)	.13725(-1)	.11523	.11523	.17211	.17211
0.6	.57347(-3)	.57347(-3)	.17183(-1)	.17183(-1)	.12744	.12744	.18790	.18790
0.7	.15128(-2)	.15128(-2)	.21680(-1)	.21680(-1)	.14010	.14010	.20364	.20364
0.8	.32437(-2)	.32437(-2)	.27331(-1)	.27331(-1)	.15319	.15319	.21933	.21933
0.9	.59604(-2)	.59604(-2)	.34166(-1)	.34166(-1)	.16667	.16667	.23496	.23496
1.0	.97712(-2)	.97712(-2)	.42142(-1)	.42142(-1)	.18047	.18047	.25049	.25049

Given a closer looking for the table 1,2 and 3, we promptly realize the very good comparison with at least five rounds positions, with shows the good computational performance of the discussed method for all the problems discussed.

4. Conclusion

The main idea of the classical method to handle radiative transfer problems in a heterogeneous plane parallel atmosphere consists in the approximation of the heterogeneous media into a multilayer slab, having the albedo coefficient in each layer an constant averaged or constant value. Once the local solution for the homogeneous slab is known, the global solution for the multilayered slab is then determined imposing the boundary condition and the interface condition of continuity for the radiation intensity at the interfaces. The proposed approach posses a new philosophy. In fact, we begin constructing the global solution, by solving the radiative transfer equation in the entire domain, assuming an averaged value for the albedo coefficient. The influence of the heterogeneity is then incorporated in the solution by solving the discussed recursive system of problems in the entire domain, with the property that the sources terms carry out the information of the heterogeneity. In fact, we construct the global heterogeneous solution progressively from the homogeneous one. Besides the genuine character of the proposed analytical solution for the S_N approach of the radiative transfer equation, as well the very good comparisons of the encountered results against the ones of the literature, we are confident to affirm that this approach is an interesting and promising methodology to work out such problems in heterogeneous media with error control, once the solution for the same problem in homogeneous media is known. We reinforce this claim recalling the proved convergence of the $\mathrm{LTS_N}$ method whe N goes to infinitive [20]. Furthermore, we need also to emphasize the generality of this solution, in the sense it is valid for all albedo coefficient functions of the spatial variable which fulfill the mathematical requirements of existence and uniqueness. By analytical we mean that no approximation is done along the solution derivation. Therefore, in this sense we may say that the solution given by equation (10) is an analytical solution of the $\mathrm{S_N}$ problem (1).

So far, bearing in mind that we pave the road to extend this solution for the nonlinear radiative transfer problem in a heterogeneous plane parallel atmosphere taking the advantage of the known solution for the same problem in homogeneous media [10], we shall focus our future attention on this direction.

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