2017 International Nuclear Atlantic Conference - INAC 2017
Belo Horizonte, MG, Brazil, October 22-27, 2017
ASSOCIAÇÃO BRASILEIRA DE ENERGIA NUCLEAR - ABEN

Agência Brasileira do ISBN
ISBN 978-85-99141-07-6

ON THE RADIATIVE-CONDUCTIVE SOLUTION IN CONTINUOUS HETEROGENEOUS GREY PLANE-PARALLEL PARTICIPATING MEDIUM

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ABSTRACT

In this work we report an analytical representation for the solution of the radiative-conductive S_N equation in a plane-parallel atmosphere in a heterogeneous domain considering an arbitrary continuous functions for the albedo. The basic idea consists in the application of the decomposition procedure to the non-linear radiative-conductive S_N problem that are easily solved by the well know LTS_N method. The length of the recursive system is properly chose in order to get a prescribed accuracy for the results. We also present numerical simulations for the results.

1. INTRODUCTION

A great variety of solutions is found in literature for the radiative transfer problem in plane parallel atmosphere either for constant albedo, as well as, for polynomial and exponential albedo function. For illustration see the works [1, 5, 6, 8, 11, 18]. However for nonlinear radiative-conductive transfer problem the literature is restricted to constant albedo [14, 16, 17, 19]. In this work we step forward reporting an analytical representation of the solution for the radiative-conductive transfer problem considering albedo as continuous function. To reach this goal following the idea of the Decomposition method, we construct a recursive system of radiative-conductive transfer equation with constant albedo, considering the contribution of spatial dependence of the albedo as source. The size of the recursive system is selected in order to get a prescribed accuracy for the results. It is worthwhile to mention that the first equation satisfies the original boundary conditions meanwhile the remaining ones satisfy the null boundary conditions. We must emphasize that the solution of all equations of the recursive system are known. From the numerical simulations performed we notice the instability of the results for this sort of problems for small values of the N_c parameter. The explanation for this behavior comes from the considered double precision for the arithmetic length chord. Therefore in this work all simulations are done for the ensuing values of the N_c parameter: 0.1 and 0.5. In addition we must recall that the mathematical analysis of existence and uniqueness of the solution for the problem discussed is found in the works [3, 4]. Without loosing generality, we specialize the application of the aforementioned solution to the polynomial albedo functions. Finally we report numerical simulations and comparisons against literature results.

2. THE RADIATIVE-CONDUCTIVE SOLUTION

In order to construct an analytical solution for the radiative-conductive problem in an inhomogeneous emitting, absorbing and grey plane-parallel slab of optical thickness discussed, let us consider the S_N problem:

$$\frac{\partial}{\partial \tau} I_n(\tau) + \frac{1}{\mu_n} I_n(\tau) - \frac{\omega(\tau)}{2\mu_n} \sum_{l=0}^{L} \beta_l P_l(\mu_n) \sum_{k=1}^{N} P_l(\mu_k) I_k(\tau) w_k = \frac{(1-\omega(\tau))}{\mu_n} \Theta^4(\tau), \qquad (1)$$

for $n = 1 : N$ and subject to the reflecting and emitting boundary condition

$$I_n(0) = \varepsilon_1 \Theta_1^4 + \rho_1^s I_{N-n+1}(0) + 2\rho_1^d \sum_{k=1}^{N/2} w_k \mu_k I_{n-k+1}(0), \qquad (1a)$$

$$I_{N-n+1}(L) = \varepsilon_2 \Theta_2^4 + \rho_2^s I_n(L) + 2\rho_2^d \sum_{k=1}^{N/2} w_k \mu_k I_k(L), \qquad (1b)$$

for $n = 1 : N/2$. Here, μ_n are the discrete directions in a decreasing order, $I_n(\tau) = I(\tau, \mu_n)$, $\tau \in [0, \tau_0]$ is the optical depth, $\omega(\tau)$ is the scattering albedo, β_l are the expansion coefficients of the phase function, P_l are the Legendre polynomials, L the highest order of polynomials and Θ is the dimensionless temperature. In the boundary conditions, ρ_i^s and ρ_i^d for $i = 1, 2$, are respectively the specular and diffuse reflections with are related to the emissivity by $1 = \varepsilon_i + \rho_i^s + \rho_i^d$, and w_k are the Gaussian weights for the approximation of the integral term using a discrete set of directions μ_k, $k = 1 : N$.

The dimensionless radiative flux is given by its relation to the intensity

$$q_r^* = 2\pi \int_{-1}^{1} I(\tau, \mu)\mu \, d\mu \approx 2\pi \sum_{k=1}^{N} I_k \mu_k w_k, \qquad (2)$$

and the energy equation for the temperature reads as

$$\frac{d^2}{d\tau^2}\Theta(\tau) = \frac{1}{4\pi N_c}\frac{d}{d\tau}q_r^*(\tau), \tag{3}$$

subject to prescribed temperatures at the boundary

$$\Theta(0) = \Theta_1 \text{ and } \Theta(\tau_0) = \Theta_2. \tag{4}$$

Here N_c is called the conduction-to-radiation parameter [12], defined by

$$N_c = \frac{k\beta_{ext}}{4\sigma n^2 T_r^3}, \tag{5}$$

where k, β_{ext}, σ, n and T_r are respectively the thermal conductivity, the extinction coefficient, the Stefan-Boltzmann constant, the refractive index and a reference temperature. The solution (3) is given analytically by:

$$\Theta(\tau) = \Theta_1 + (\Theta_2 - \Theta_1)\frac{\tau}{\tau_0} - \frac{1}{4\pi N_c}\frac{\tau}{\tau_0}\int_0^{\tau_0} qr^*(\tau')d\tau' + \frac{1}{4\pi N_c}\int_0^{\tau} qr^*(\tau')d\tau'. \tag{6}$$

Now, we recall that the equation (1) relates the intensity of radiation to temperature, that equation (3) shows the connection between temperature and radiative flux and finally that equation (2) associates the flux with the intensity of radiation. These facts allow us to convert the problem into a form that depends only on the directional intensity $I_n(\tau)$.

in order to apply the decomposition method to problem (1), we split the albedo function as $\omega(\tau) = \bar{\omega} + \omega_0(\tau)$, where $\bar{\omega}$ is the average albedo function [18] and we expand the non-linear source term into a series of Adomian polynomials [2] as,

$$\Theta^4(\tau) = \sum_{m=0}^{\infty} \widehat{A}_m(\tau). \tag{7}$$

Upon inserting this ansatz in equation (1) yields a first order matrix differential equation:

$$\frac{d}{d\tau}\mathbf{I}(\tau) - \mathbf{A}\ \mathbf{I}(\tau) = \mathbf{S}(\tau) + (1 - \omega(\tau))\sum_{m=0}^{\infty} \hat{A}_m(\tau)\mathbf{M}. \tag{8}$$

Here, $\mathbf{I}(\tau) = \text{col}[\mathbf{I}_1(\tau) \quad \mathbf{I}_2(\tau)]$ is the intensity radiation vector, where the subvectors $\mathbf{I}_1(\tau)$ and $\mathbf{I}_2(\tau)$ are the intensity radiation for the positive $(0 < \mu < 1)$ and negative

$(-1 < \mu < 0)$ directions, respectively, and of order $N/2$ each. Further, $\mathbf{M}$ and $\mathbf{S}(\tau)$ are vectors of order N, whose the n^{th} entries are defined as $m_n = \frac{1}{\mu_n}$ and

$$s_n(\tau) = \frac{\omega_0(\tau)}{2\mu_n} \sum_{l=0}^{L} \beta_l P_l(\mu_n) \sum_{k=1}^{N} P_l(\mu_k) I_k(\tau) w_k. \tag{9}$$

Finally, the components of matrix $\mathbf{A}$ have the form:

$$a_{ij} = -\frac{1}{\mu_i}\delta_{ij} + \frac{\varpi}{2\mu_i} \sum_{l=0}^{L} \beta_l P_l(\mu_i) P_l(\mu_j) w_k, \quad \delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} . \tag{10}$$

where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ for $i = j$ and 0 otherwise). The radiation intensity can formally be written as

$$\mathbf{I}(\tau) = \sum_{m=0}^{\infty} \mathbf{I}^m(\tau), \tag{11}$$

with upon decomposition substitution in equation (8) results

$$\sum_{m=0}^{\infty} \left(\frac{d}{d\tau}\mathbf{I}^m(\tau) - \mathbf{A}\ \mathbf{I}^m(\tau) \right) = \sum_{m=0}^{\infty} \Big(\mathbf{S}^{m-1}(\tau) + (1-\omega(\tau))\hat{A}_{m-1}(\tau)\mathbf{M} \Big) . \tag{12}$$

The solution of this equation is undetermined because we have one equation with many unknowns vector functions $\mathbf{I}^{m-1}(\tau)$. Therefore the solution is not unique. In this work we solve this problem choosing the following recursive system of equations:

$$\begin{aligned} &\tfrac{d}{d\tau}\mathbf{I}^0(\tau) - \mathbf{A}\mathbf{I}^0(\tau) = 0, \\ &\tfrac{d}{d\tau}\mathbf{I}^m(\tau) - \mathbf{A}\mathbf{I}^m = \mathbf{S}^{m-1}(\tau) + (1-\omega(\tau))\widehat{A}_{m-1}(\tau)\mathbf{M}, \quad m = 1, 2 ... \mathcal{M}, \end{aligned} \tag{13}$$

which is then solved by the LTS$_N$ method [15, 7, 13] for any arbitrary but finite $m \leq \mathcal{M}$. Here $\mathcal{M}$ is a truncation of the series which is chosen to get a prescribed accuracy. The motivation for this choice comes from the fact that all equations in the recursive system have know analytical representation for the solution. Furthermore, we consider that initial equation ($m = 0$) satisfies the boundary condition given by equations (1a,1b) meanwhile the remaining equations the homogeneous boundary conditions.

The LTS$_N$ solution of the homogeneous equation of the recursive system (13) is given by

$$\mathbf{I}^0(\tau) = \mathbf{X}\mathbf{E}^{\mathbf{D}(\tau)}\mathbf{V}^0, \tag{14}$$

where $\mathbf{D}$ and $\mathbf{X}$ are respectively the eigenvalues and eigenvectors matrices resulting from the spectral decomposition of $\mathbf{A}$ matrix. The $\mathbf{E}^{\mathbf{D}(\tau)}$ matrix is defined as:

$$\mathbf{E}^{\mathbf{D}(\tau)} = \begin{cases} e^{d_{ii}\tau} & if \quad d_{ii} < 0 \\ e^{d_{ii}(\tau-\tau_0)} & if \quad d_{ii} > 0 \end{cases}, \tag{15}$$

where d_{ii} are entries of the eigenvalue matrix $\mathbf{D}$. Further, the general solution for the remaining problems of the recursive system are given as

$$\mathbf{I}^m(\tau) = \mathbf{X}\mathbf{E}^{\mathbf{D}(\tau)}\mathbf{V}^m + \mathbf{X}\mathbf{e}^{\mathbf{D}\tau}\mathbf{X}^{-1} * (\mathbf{S}^{m-1}(\tau) + (1-\omega(\tau))\widehat{A}_{m-1}(\tau)\mathbf{M}), \tag{16}$$

where $\mathbf{e}^{\mathbf{D}\tau}$ is the exponential matrix function, the star $*$ denotes the convolution operator and the vectors $\mathbf{V}^m$ are determined from boundary conditions for each problem. We must note that the boundary conditions of the original problem is absorbed by the recursive system in the first recursion, whereas the remaining problems satisfy the homogeneous boundary conditions.

The role of the Adomian polynomials is the approximation of the non-linear term in eq. (1), i.e. the dimensionless non-linear temperature term Θ^4. Using a finite functional expansion in $T_m(\tau)$ for the dimensionless temperature, $\Theta = \sum_{m=0}^{\mathcal{M}} T_m(\tau)$. For simplicity, we must emphasize that the polynomials $\widehat{A}_m(\tau)$ are determined by a simple recursive formula developed by Segatto et al [14] as follow

$$\widehat{A}_m = T_m S_m R_m, \tag{17}$$

where S_m and R_m are determined by the formulas

$$\begin{aligned} S_m &= S_{m-1} + T_m + T_{m-1}, \\ R_m &= R_{m-1} + S_{m-1}T_{m-1} + S_m T_m, \end{aligned} \tag{18}$$

with $S_0 = T_0$ and $R_0 = T_0^2$ and

$$T_0(\,\tau\,) = \theta_1 + (\theta_2 - \theta_1)\frac{\tau}{\tau_0} - \frac{1}{2N_c}\frac{\tau}{\tau_0}\sum_{k=1}^{N}\mu_k w_k \int_0^{\tau_0} I_k^0(\tau')d\tau' + \frac{1}{2N_c}\sum_{k=1}^{N}\mu_k w_k \int_0^{\tau} I_k^0(\tau')d\tau' \tag{19a}$$

$$T_{m+1}(\tau) = -\frac{1}{2N_c}\frac{\tau}{\tau_0}\sum_{k=1}^{N}\mu_k w_k \int_0^{\tau_0} I_k^m(\tau')d\tau' + \frac{1}{2N_c}\sum_{k=1}^{N}\mu_k w_k \int_0^{\tau} I_k^m(\tau')d\tau'. \tag{19b}$$

Here $m = 0, 1, ..., \mathcal{M}$. Note that equations (19) establish the Adomian polynomial in terms of the temperature at the boundaries and the expansion terms of the intensity, with in principle could be determined until infinity.

3. NUMERICAL RESULTS

Aiming to show the aptness of the proposed method to solve the coupled conductive-radiative heat transfer problems in a slab with space-dependent albedo coefficient, in the sequel we report numerical results for the following problems:

Table 1: Parameters used in the tables and figures.

ε_1	ε_2	ρ_1^s	ρ_2^s	ρ_1^d	ρ_2^d	θ_1	θ_2	τ_0	L
1	1	0	0	0	0	1	0	1	0

Where, $\varepsilon_i = 1$, $i = 1, 2$ are emissivities, $\rho_i^s = \rho_i^d = 0$, $i = 1, 2$ are the coefficients for specular and diffuse reflections respectively, $\theta_i, i = 1, 2$ the dimensionless temperature and τ_0 maximum optical depth.

Initially, we solve the radiative-conductive transfer problem in planar geometry with the parameters depicted in Table 1.

Due the lack of results in literature for this sort of problem, we focus our attention in analysis of validation of the results for a linear polynomial albedo. To reach this goal this problem is solved for a set of values for the coefficient of the linear polynomial term going to zero, namely $a = 0.1, 0.01, 0.001, 0.0001$. The results for the albedo $\omega = a\tau + 0.8$ is shown in Table 2. We compare the results attained against the ones for constant albedo. Given a closer look to Table 2, we promptly realize a good coincidence of five significant digits between the linear polynomial and constant for a specific value of $a = 0.0001$. By this procedure we are confident to affirm that we have shown that the solution for polynomial albedo considered goes to the value of the constant albedo when the coefficient, a, goes from 0.1 to 0.0001.

Table 2: Solution of the recursive system for the radiation intensity in $\tau = 0.5$ and $N_c = 0.5$ and albedo functions.

μ	$\omega = 0.8$	$\omega(\tau) = 0.0001\tau + 0.8$	$\omega(\tau) = 0.001\tau + 0.8$	$\omega(\tau) = 0.01\tau + 0.8$	$\omega(\tau) = 0.1\tau + 0.8$
-1	0.098129	0.098143	0.098271	0.099560	0.113403
-0,5	0.162339	0.162362	0.162567	0.164637	0.186860
-0.001	0.340748	0.340780	0.341069	0.343976	0.375037
0.001	0.341749	0.341781	0.342069	0.344967	0.375927
0,5	0.676022	0.676034	0.676142	0.677227	0.688865
1	0.804963	0.804970	0.805033	0.805665	0.812446

Next we solve the coupled conductive-radiative heat transfer problem in planner geometry with the purpose to determine the size of the recursive system, we mean the number of equations solved ($\mathcal{M}$), in order to get a prescribed accuracy, $\varepsilon = 10^{-6}$. For such, we solve the same conductive-radiative heat transfer problem but now for second order polynomial albedo coefficient, as well, $N_c = 0.5$. In Table 3, we present the results achieved.

Table 3: Recursive system solution for the radiation intensity at $\tau = 0.5$ considering $\omega(\tau) = 1 - 1.4\tau - 0.6\tau^2$ and $N_C = 0.1$ from $\mathcal{M}$ varying from 1 to 14.

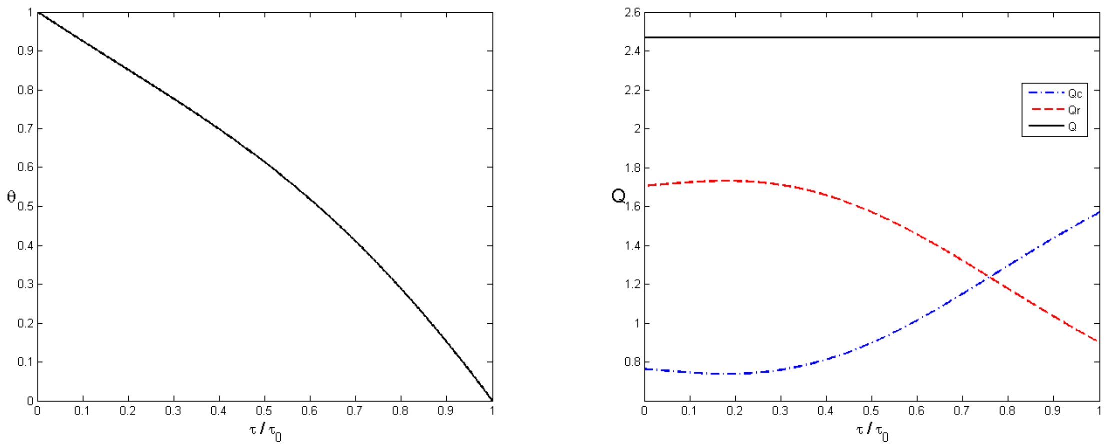
μ	D_1LTS_{100}	D_2LTS_{100}	D_3LTS_{100}	D_4LTS_{100}	D_5LTS_{100}	D_6LTS_{100}	D_7LTS_{100}
-1	0,048076	0,036649	0,040911	0,039506	0,039943	0,039809	0,039850
-0,5	0,083462	0,063950	0,071163	0,068798	0,069533	0,069307	0,069376
-0.001	0,275484	0,217999	0,236537	0,230849	0,232579	0,232053	0,232213
0.001	0,277566	0,219739	0,238334	0,232634	0,234367	0,233840	0,234000
0,5	0,663267	0,639634	0,647841	0,645664	0,646334	0,646134	0,646195
1	0,796315	0,783779	0,788512	0,787295	0,787673	0,787560	0,787594
μ	D_8LTS_{100}	D_9LTS_{100}	$D_{10}LTS_{100}$	$D_{11}LTS_{100}$	$D_{12}LTS_{100}$	$D_{13}LTS_{100}$	$D_{14}LTS_{100}$
-1	0,039837	0,039841	0,039840	0,039840	0,039840	0,039840	0,039840
-0,5	0,069355	0,069361	0,069359	0,069360	0,069360	0,069360	0,069360
-0.001	0,232164	0,232179	0,232175	0,232176	0,232176	0,232176	0,232176
0.001	0,233951	0,233966	0,233962	0,233963	0,233963	0,233963	0,233963
0,5	0,646176	0,646182	0,646180	0,646181	0,646180	0,646180	0,646180
1	0,787584	0,787587	0,787586	0,787587	0,787586	0,787586	0,787586

We readily realize from the table above that the results encountered by the method $D_{13}LTS_{100}$ possess an accuracy of 10^{-6}. This affirmative is supported by the coincidence of six significant digits between $D_{13}LTS_{100}$ and $D_{14}LTS_{100}$ results. Furthermore, after the previous analysis we also show numerical simulations in Tables 4 and 5 for the normalized conductive, radiative and total heat flux using Equations (20).

$$Q_r(\tau) = \frac{1}{4\pi N_c} q_r^* \qquad Q_c(\tau) = -\frac{d}{d\tau}\theta(\tau) \qquad Q(\tau) = Q_r(\tau) + Q_c(\tau). \tag{20}$$

Table 4: Numerical results $D_{13}LTS_{100}$ with $\omega(\tau) = 0.95 - 0.9\tau$ and $Nc = 0.1$ using data of the table 1.

τ	θ	Qc	Qr	Q
0	1.000000	0.762977	1.705941	2.468918
0.25	0.813569	0.744530	1.724388	2.468918
0.5	0.612543	0.898406	1.570512	2.468918
0.75	0.349824	1.221526	1.247392	2.468918
1	0.000000	1.568781	0.900137	2.468918

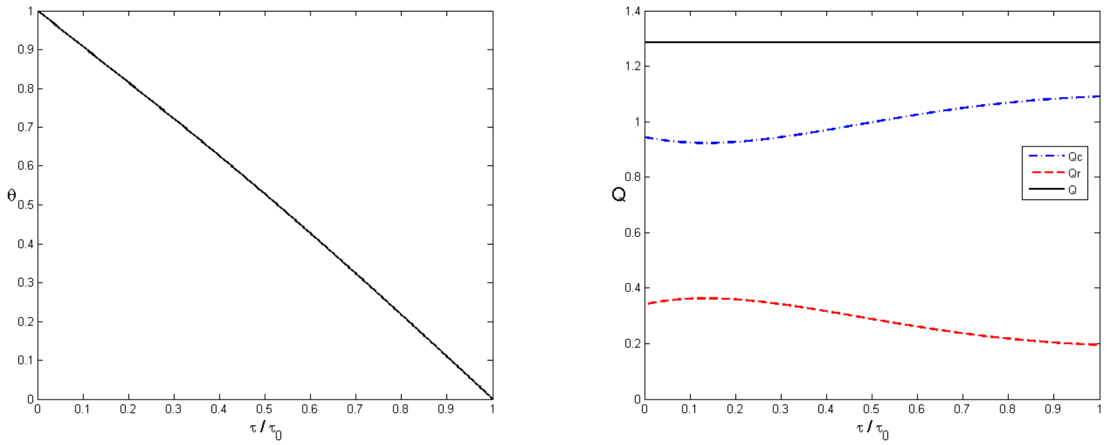


(a) Temperatura (b) Fluxos condutivo, radiativo e total

Figure 1: Graph of the Conductive flows, radiative and total for $D_{13}LTS_{100}$ with $\omega(\tau) = 0.95 - 0.9\tau$ and $Nc = 0.1$ using data of the table 1.

Table 5: Numerical results for $D_{13}LTS_{100}$ with $\omega(\tau) = 0.4 - 0.2\tau + 0.6\tau^2$ and $N_c = 0.5$ using data of the table 1.

τ	θ	Qc	Qr	Q
0	1.000000	0.944760	0.340896	1.285656
0.25	0.767899	0.933951	0.351705	1.285656
0.5	0.526992	0.997648	0.288008	1.285656
0.75	0.269399	1.059131	0.226525	1.285656
1	0.000000	1.090690	0.194966	1.285656



(a) Temperatura (b) Fluxos condutivo, radiativo e total

Figure 2: Graph of the Conductive flows, radiative and total for $D_{13}LTS_{100}$ com $\omega(\tau) = 0.4 - 0.2\tau + 0.6\tau^2$ e $Nc = 0.5$ using data of the table 1.

4. CONCLUSIONS

In this work we present a genuine hierarchical analytical representation for the solution of the radiative-conductive transfer equation with arbitrary albedo continuous function once the albedo function satisfies the required conditions for existence. By hierarchical we mean that the solution of the nonlinear radiative-conductive transfer problem is determined from the knowledge of the solution of the linear radiative transfer problem. Furthermore, by analytical we mean that no approximation is done along the solution derivation except for the choice of the length of the recursive system. We must underline that the accuracy of the results obtained is determined by a suitable choice of the length of the recursive system, that means, the number of equations of recursive system solved. To complete the mathematical analysis of the proposed solution two open questions must be answered. To reach this goal, following the idea of the Lyapunov theory for the analysis of convergence we shall show an heuristic analysis of convergence for the solution of the recursive system of the considered problem. On the other hand, following the idea of the decomposition procedure discussed in the work of Ladeia et al [9, 10], we also will construct a stable solution of the radiative-conductive S_N problem solved for all values for the N_c parameters. We focus our future attention in this direction.

ACKNOWLEDGMENTS

The authors MTV e RMFV thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) for the partial financial support of this work and FLV thanks IFRS (Instituto Federal de Educação Ciência e Tecnologia do Rio Grande do Sul) for the financial support and absence from work.

REFERENCES

1. E.M. Abulwafa: "Condutive-radiative heat transfer in an inhomogeneous slab with directional reflecting boundaries". *Journal of Physics* D, **32**, pp. 1626-1632, (1999).
2. G. Adomian: "A Review of the Decomposition Method in Applied Mathematics".*journal mathematical analysis application* **135**, pp. 501-544, (1988).
3. F.S. Azevedo, E. Sauter, M. Thompson, M.T. Vilhena: "Existence theory and simulations for one-dimensional radiative flows". *Annals of Nuclear Energy*, **38**, p. 1115-1124, (2011).
4. F.S. Azevedo, E. Sauter, M. Thompson, M.T. Vilhena: "Existence theory for a one-dimensional problem arising from the boundary layer analysis of radiative flows" *Progress in Nuclear Energy (New Series),* **53**, p. 1105-1113, (2011).
5. Y.A. Cengel, M.N. Ozisik: "Radiative Transfer in Plane Parallel Medium with Space-Dependent Albedo $\omega(x)$". *International Journal of Heat and Mass Transfer*, **27**, pp. 1919-1922, (1984).
6. R.D.M. Garcia, C.E. Siewert, "Radiative Transfer in Finite Inhomogeneous Plane-Parallel atmospheres." *J of Quantitative Spectroscopy and Radiative Transfer*, **vol. 27**, **No 2**, pp. 141-148, (1982).

7. G.A. Gonçalves, M.T. Vilhena, "The LTS_N Particular Solution in a Slab for an Abritrary Source and Large Order of Quadrature". *Journal of Quantitative Spectroscopy an Radiative Transfer*, **66**, pp 271-276, (2000).
8. A.E. Kovtangyuk, N.D. Botkin, K.H. Hoffmann: "Numerical simulations of a coupled radiative-conductive heat transfer model using a modified Monte Carlo method". *Int J of Heat and Mass Transfer,* **55**, pp. 649-654, (2012).
9. C.A. Ladeia, B.E.J. Bodmann, M.T. Vilhena: "The Radiative-Conductive Transfer Equation in Cylindrical Geometry". *Journal of Computational and Theoretical Transport*, **45**, p. 1-10, (2016).
10. C.A. Ladeia, B.E.J. Bodmann, M.T. Vilhena: "The Radiative-Conductive Transfer Equation in Cylinder Geometry and Its Application to Rocket Launch Exhaust Phenomena", in Întegral Methods in Science and Engineering. Springer International Publishing, 2015, p. 341-351.
11. J. Ma, Y.S. Sun, B.W. Li: "Analysis of radiative trasfer in a one-dimensional nonlinear anisotropic scattering medium with space-dependent scatterint coeficiente using spectral collocation method". *International Journal of Heat and Mass Transfer,* **67** pp. 569-574, (2013).
12. M.N. Ozisik: *Radiative Transfer and Interactions with Conduction and Convection.* Primeira edição, John Wiley & Sons, New York, USA, (1973).
13. R.P. Pazos, M.T. Vilhena, "Convergence of the LTS_N Method: Approach of semigroups". *Progress in Nuclear Energy*, **30**, pp. 77-86, (1999).
14. C.F. Segatto, R.M.F. Vargas, M.T. Vilhena, B.E.J. Bodmann: "A solution for the non-linear S_N radiative-condutive problem in a grey plane-parallel participant medium". *International Journal of Thermal Sciences*, **49**, pp. 1493-1499, (2010).
15. C.F.Segatto, M.T. Vilhena, M.G. Gomes, "The one-dimentional LTS_N solution in a slab with high defree of quadrature." *Annals of Nuclear Energy*, **26**, pp. 925-934, (1999).
16. C.E. Siewert, J.R. Thomas Jr.: "A computational method for solving a class of coupled condutive-radiative heat transfer problems". *Journal of Quantitative Spectroscopy and Radiative Transfer*, **45**, pp 599-605, (1995).
17. C.E. Siewert,: "An improved iterative method form solving a class of coupled condutive-radiative heat transfer problems". *Journal of Quantitative Spectroscopy and Radiative Transfer*, **54**, pp. 273-281, (1991).
18. R.M.F. Vargas, C.F.Segatto, M.T. Vilhena: "On the Analytical Solution of the S_N Radiative Transport Equation in a Slab for a Space-dependent Albedo Coefficient." *Journal of Physics (conference series)*, **369**, (2012).
19. Y. Zhang, H.L. Yi, H.P. Tan: "Natural elemente method analysis for coupled radiative and condutive heat transfer in semitransparent medium with irregular geometrias." *International Journal of Thermal Sciences*, **76**, pp 30-42, (2014).