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Controller and anti-windup co-design for the output regulation of rational systems subject to control saturation

Rafael S. Castro[1] | **Jeferson V. Flores**[2] | **Aurélio T. Salton**[2] | **João M Gomes da Silva Jr.**[2]

[1]School of Technology, Pontifical Catholic University of Rio Grande do Sul, Porto Alegre, Brazil

[2]School of Engineering, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil

Correspondence
Rafael S. Castro, School of Technology, Pontifical Catholic University of Rio Grande, do Sul, Av. Ipiranga 6681, Porto Alegre, Rio Grande do Sul 90619-900, Brazil.
Email: rafael.castro@pucrs.br

Abstract

This article presents a novel framework for the output regulation of rational nonlinear systems subject to input saturation, where the controller structure is composed by an internal model generator in series with an output feedback stabilizing stage. In order to address the effects of control saturation, we propose the use of an anti-windup compensation loop into both internal model and stabilizing control stages. Given prior knowledge of the system zero-error steady-state condition and a proper internal model, we cast the regulation error dynamics in a differential-algebraic form, which leads to matrix inequalities conditions that ensure closed-loop stability and exponential transient performance. Optimization problems are then proposed to simultaneously compute the stabilizing controller and the anti-windup gains such that a domain of attraction estimate is maximized.

KEYWORDS

anti-windup, input saturation, linear matrix inequalities, nonlinear systems, output regulation

1 | INTRODUCTION

Output regulation is a widely studied control problem that deals with dynamical systems subject to nonvanishing disturbance and reference signals produced by an exogenous autonomous system.[1] In the context of linear systems, the output regulation problem was primarily solved by Francis and Wonham,[2] who proposed the classical internal model principle. On the other hand, the theory on output regulation of nonlinear systems started to develop with References 3 and 4, where the zero-error steady-state conditions (so-called regulator equations) were introduced and guidelines were proposed to tackle a nonlinear regulator design problem in order to achieve closed-loop stabilization. Later developments have contributed with design methods for special classes of nonlinear regulation problems. For example, Reference 5 proposed an adaptive error feedback scheme for a class of minimum phase uncertain nonlinear systems while Reference 6 proposed ways to design nonlinear internal models in order to address systems with nonpolynomial nonlinearities. More recently, Reference 7 showed a method for output regulation with exponential convergence properties, while Reference 8 dealt with nonlinear cascaded systems with integral dynamic uncertainties. It should be pointed out that these methods are generally restricted to input-affine nonlinear systems representable in a triangular form. Moreover, an open problem is to systematically address a saturation nonlinearity involving the control input and also to explore anti-windup design schemes in the nonlinear output regulation context.

[Correction added on 12 January 2021, after first online publication: in the article title, ‘code-sign’ was changed to ‘co-design’, and the ORCID link of the author Salton was added in this version. Also, in the text, ‘codesign’ and ‘antiwindup’ were changed to ‘co-design’ and ‘anti-windup’, respectively.]

With the development of interior point solvers, many analysis and design methods for linear systems started to appear in the form of optimization problems subject to linear matrix inequalities (LMIs).[9] There have been efforts in order to extend these methodologies for classes of nonlinear systems, such as the differential-algebraic representation (DAR).[10] The main purpose of this method is to deal with rational nonlinearities (products and quotients of polynomial functions), which may be found in many real-world applications such as spacecraft control,[11] unmanned aerial vehicles,[12] robotic manipulators,[13] and chaos-based cryptography.[14] DAR-based methodologies have been extensively investigated in many different scenarios with several improvements and extensions. Initially, a method for stabilization and domain of attraction estimation was developed for rational nonlinear systems,[15] and was subsequently adapted for systems subject to input saturation.[16] Later on, a similar design approach was proposed for the input-to-state stabilization problem in the presence of actuator saturation[17] and methods were also devised for the static and dynamic anti-windup stabilization problems.[18,19] Moreover, Reference[10] brought a complete overview of the differential-algebraic theory, focusing on criteria for local, regional and global asymptotic stability of uncertain rational nonlinear systems. More recently, the DAR method was proposed for the stability analysis of output regulation control loops subject to rational error dynamics.[20] This result was later generalized into a controller design framework for closed-loop stabilization of rational nonlinear output regulation schemes with internal model stages.[21] In order to extend the scope of the aforementioned study, it is possible to address the control input saturation effect and to investigate the use anti-windup compensation, which are the subject of the present work.

In this article, we present a novel framework for the output regulation of rational systems subject to control input saturation. The control scheme proposed here unifies the classical internal model-based control for nonlinear systems[22] and the anti-windup compensation discussed in References 18,23, both well-established approaches which have not been explored together yet. The resulting control architecture is thus composed by a stabilizing dynamic output feedback controller in series with an internal model structure, where anti-windup compensation is considered in both stages so as to mitigate the undesired effects arising from input saturation. To tackle the problem of closed-loop stabilization subject to rational nonlinearities and input saturation, we propose a new modified sector condition combined with a DAR approach. The co-design of the stabilizing controller parameters and the anti-windup gains are then expressed as optimization problems aiming to maximize a domain of attraction estimate. A special design case is also shown to lead a convex optimization problem based on LMI constraints. As a result, our framework is systematic and applicable to a large class of output regulation problems, namely, cases whose regulation error dynamics may be described by rational nonlinear functions and where the control input is possibly subject to saturation, therefore including problems that are not directly tractable by state-of-the-art nonlinear output regulation methods as previously discussed.

Notation: x_i is the ith element of vector x. $A_{[i]}$ is the ith row of matrix A. $A_{[i,j]}$ denotes a term located at the ith row and jth column of matrix A. The transpose of matrix A is represented by $A^{\top}$. $A > 0$ means that matrix A is positive-definite. $\mathrm{He}\{A\} = A + A^{\top}$. $\mathrm{diag}\{A, B\}$ denotes a diagonal matrix obtained by A and B. $\mathrm{tr}(A)$ is the trace of A. $(\star)$ represents symmetric elements in a matrix, whereas $(\cdot)$ hides irrelevant terms. $\mathcal{V}\{\mathcal{X}\}$ denotes the vertices of a polytope $\mathcal{X}$.

2 | PRELIMINARIES

Consider a nonlinear system represented by

$$\begin{cases} \dot{x} = f(x, w, u) \\ y = g(x, w) \\ e = h(x, w) \end{cases}, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $y \in \mathbb{R}^{n_y}$ is the output measurement and $e \in \mathbb{R}^{n_e}$ is the output error. The system control input $u \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ is supposed to be constrained to the set

$$\mathcal{U} = [-\overline{u}_1, \overline{u}_1] \times [-\overline{u}_2, \overline{u}_2] \times \dots \times [-\overline{u}_{n_u}, \overline{u}_{n_u}], \tag{2}$$

where $\overline{u}_1, \dots, \overline{u}_{n_u} > 0$. System (1) is also disturbed by an exogenous signal $w \in \mathbb{R}^{n_w}$ generated by a nonlinear exosystem:

$$\dot{w} = s(w). \tag{3}$$

Functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathcal{U} \to \mathbb{R}^{n_x}$, $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_y}$, $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_e}$ and $s : \mathbb{R}^{n_w} \to \mathbb{R}^{n_w}$ are nonlinear and satisfy $f(0,0,0) = 0$, $g(0,0) = 0$, $h(0,0) = 0$, and $s(0) = 0$. The control input is provided by a nonlinear output feedback controller:

$$\begin{cases} \dot{\xi} = \phi(\xi, y) \\ u = \theta(\xi, y) \end{cases}, \tag{4}$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the controller state vector and $\phi : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_\xi}$, $\theta : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_y} \to \mathcal{U}$ are nonlinear functions satisfying $\phi(0,0) = 0$ and $\theta(0,0) = 0$. Given this preliminary setup, the following basic definitions and assumptions are considered.

Definition 1. The trajectories of the closed-loop system described by (1), (3), (4) are said to be bounded if $\exists \epsilon_1, \epsilon_2, \epsilon_3 > 0 : \|x(t)\| \leq \epsilon_1, \|\xi(t)\| \leq \epsilon_2, \|w(t)\| \leq \epsilon_3 \forall t \geq 0, \forall (x(0), \xi(0), w(0)) \in \mathcal{D}$, for some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w}$.

Definition 2. The trajectories of the closed-loop system described by (1), (3), (4) are said to achieve output regulation in some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w}$ if they are *bounded* and, moreover, $\lim_{t\to\infty} \|e(t)\| = 0 \ \forall (x(0), \xi(0), w(0)) \in \mathcal{D}$.

Assumption 1. There exist a known compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$ such that $w(t) \in \mathcal{W} \ \forall \ t > 0$ if $w(0) \in \mathcal{W}$.

The main control design problem to be dealt in this article can be described as follows.

Problem 1. Design controller functions $\phi(\xi, y)$ and $\theta(\xi, y)$ such that the trajectories of the closed-loop system (1), (3), (4) achieve output regulation in some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathcal{W}$ with the control input signal restricted to the set (2).

In order to provide a solution for Problem 1, one must consider the following fundamental lemma which states sufficient conditions for the output regulation.[3]

Lemma 1. 3 *The trajectories of the closed-loop system (1), (3), (4) achieves output regulation in $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathcal{W}$ if there exist smooth mappings $\pi : \mathcal{W} \to \mathbb{R}^{n_x}$, $\sigma : \mathcal{W} \to \mathbb{R}^{n_\xi}$, $c : \mathcal{W} \to \mathcal{U}$ and $d : \mathcal{W} \to \mathbb{R}^{n_y}$ such that $\pi(0) = 0$, $\sigma_m(0) = 0$, $c(0) = 0$, $d(0) = 0$,*

$$\begin{cases} \frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, c(w)) \\ d(w) = g(\pi(w), w) \\ 0 = h(\pi(w), w) \end{cases} \quad \forall \ w \in \mathcal{W}, \tag{5}$$

$$\begin{cases} \frac{\partial \sigma(w)}{\partial w} s(w) = \phi(\sigma(w), d(w)) \\ c(w) = \theta(\sigma(w), d(w)) \end{cases} \quad \forall \ w \in \mathcal{W}, \tag{6}$$

and also

$$\begin{cases} \lim_{t\to\infty} \|x(t) - \pi(w(t))\| = 0 \\ \lim_{t\to\infty} \|\xi(t) - \sigma(w(t))\| = 0 \end{cases} \quad \forall \ (x(0), \xi(0), w(0)) \in \mathcal{D}. \tag{7}$$

In order to Problem 1 be solvable based on Lemma 1, we assume there exist known solutions $\pi(w)$, $d(w)$, and $c(w)$ with respect to condition (5). Moreover, the control input limits $\overline{u}_1, \dots, \overline{u}_{n_u}$ are considered to be satisfied by $c : \mathcal{W} \to \mathcal{U}$, that is, the available control amplitude is sufficient to generate the signal $c(w)$ required to achieve output regulation in steady-state, as formalized in Assumption 2. To additionally ensure the stabilization problem tractability with the DAR approach,[20] we initially consider that all system functions are regionally regular rational[1], as in Assumption 3.

Assumption 2. There exist known mapping functions $\pi(w)$, $d(w)$, and $c(w)$ satisfying (5) and

$$\sup_{w\in\mathcal{W}} |c_j(w)| < \overline{u}_j \ \forall \ j \in \{1, 2, \dots, n_u\}. \tag{8}$$

[1]Regular rational functions are those that can be expressed as fraction of polynomial functions and that has no singularities in their domain.

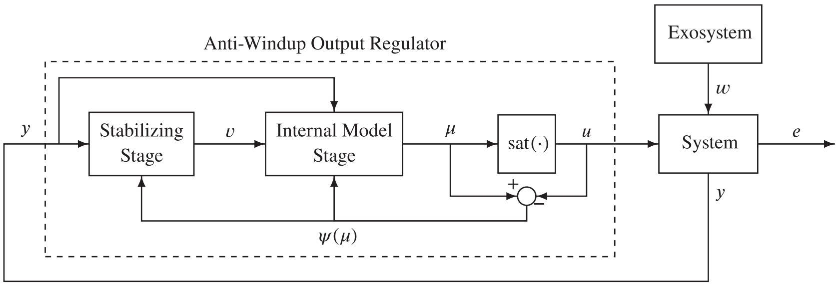

FIGURE 1 Anti-windup output regulator scheme

Assumption 3. Functions $f(x, w, u)$, $g(x, w)$, $h(x, w)$, $s(w)$, $\phi(\xi, y)$, $\theta(\xi, y)$, $\pi(w)$, $\sigma(w)$, $c(w)$, and $d(w)$ are regular rational with respect to some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \times \mathcal{W}$.

3 | OUTPUT REGULATION FRAMEWORK

This section presents the proposed output regulation framework in order to systematically address the Problem 1 stated above. We initially detail the structure of the proposed output regulator, which is composed by an internal model stage, a dynamic stabilizing controller and two anti-windup loops. The main results are presented later, where stability and performance conditions, in the form of matrix inequalities, are formulated for the simultaneous design of the controller and anti-windup parameters. This section also contains an illustrative numerical example to illustrate each development step-by-step.

3.1 | Output regulator and steady-state conditions

Our control framework is as depicted by the block diagram of Figure 1. This structure is inspired by two classical control approaches: the internal model-based control[22] and the anti-windup stabilization strategy.[18,23] As seen in Figure 1, the control input $u \in \mathcal{U}$ is primarily generated by a saturation function of an unconstrained control signal $\mu \in \mathbb{R}^{n_u}$, such as

$$u = \text{sat}(\mu), \tag{9}$$

which is defined according to

$$u_j = \text{sat}(\mu_j) \triangleq \min\{\max\{\mu_j,\ -\overline{u}_j\},\ \overline{u}_j\}\ \forall\, j \in \{1, 2, \dots, n_u\}. \tag{10}$$

This unconstrained control signal μ is in turn produced by an internal model stage, which is responsible for generating the target zero-error steady-state related to conditions (5) and (6). This last block receives a signal $v \in \mathbb{R}^{n_v}$ from a dynamic stabilizing stage, responsible for ensuring the convergence of plant and controller states to the steady-state manifold by mappings $\pi(w)$ and $\sigma(w)$ in Lemma 1. Two anti-windup loops that consists in feeding both the internal model and stabilizing controller with the control input deadzone signal, defined as

$$\psi(\mu)\ \triangleq\ \mu - u = \mu - \text{sat}(\mu), \tag{11}$$

are considered to mitigate the saturation effects in the output regulation scheme. Note that $\psi(\mu_j) = 0$ whenever $|\mu_j| \le \overline{u}_j$, whereas $\psi(\mu_j) \ne 0$ in cases that $|\mu_j| > \overline{u}_j$.

The equations for the internal model stage shown in the block diagram of Figure 1 are given by

$$\begin{cases} \dot{\xi}_m = \phi_m(\xi_m, y) + v_m + \mathrm{E}(y)\ \psi(\mu) \\ \mu = \theta_m(\xi_m, y) + v_u \end{cases}, \quad \begin{bmatrix} v_u \\ v_m \end{bmatrix} \triangleq v, \tag{12}$$

where $\xi_m \in \mathbb{R}^{n_m}$ is the internal model state vector, $v_m \in \mathbb{R}^{n_m}$ is the internal model stabilizing input and $v_u \in \mathbb{R}^{n_u}$ is the plant stabilizing input. Nonlinear functions $\phi_m : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_m}$ and $\theta_m : \mathbb{R}^{n_m} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ here define the internal model dynamics, whereas $\mathrm{E} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_m \times n_u}$ is the internal model anti-windup gain matrix. Similarly, the stabilizing stage dynamics is defined by

$$\begin{cases} \dot{\xi}_s = \phi_s(\xi_s, y) + \mathrm{W}(y)\ \psi(\mu) \\ v = \theta_s(\xi_s, y) \end{cases}, \tag{13}$$

where $\xi_s \in \mathbb{R}^{n_s}$ is the stabilizing controller state vector, $\phi_s : \mathbb{R}^{n_s} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_s}$ and $\theta_s : \mathbb{R}^{n_s} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ are the stabilizing controller functions and $\mathrm{W} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_u}$ is the stabilizing controller anti-windup gain matrix.

Based on Lemma 1, the following lemma provides sufficient output regulation conditions under the use of the anti-windup compensated control stages (12) and (13).

Lemma 2. *The trajectories of the closed-loop system composed by (1), (3) and controller (9), (12), (13) achieves output regulation in* $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_s} \times \mathcal{W}$ *if there exist smooth mappings* $\pi : \mathcal{W} \to \mathbb{R}^{n_x}$, $\sigma_m : \mathcal{W} \to \mathbb{R}^{n_m}$, $c : \mathcal{W} \to \mathcal{U}$ *and* $d : \mathcal{W} \to \mathbb{R}^{n_y}$ *satisfying* $\pi(0) = 0$, $\sigma_m(0) = 0$, $c(0) = 0$, $d(0) = 0$, *relations in (5), (8),*

$$\begin{cases} \frac{\partial \sigma_m(w)}{\partial w} s(w) = \phi_m(\sigma_m(w), d(w)) \\ c(w) = \theta_m(\sigma_m(w), d(w)) \end{cases} \quad \forall\, w \in \mathcal{W}, \tag{14}$$

$$\begin{cases} 0 = \phi_s(0, d(w)) \\ 0 = \theta_s(0, d(w)) \end{cases} \quad \forall\, w \in \mathcal{W}, \tag{15}$$

and also

$$\begin{cases} \lim_{t\to\infty} \|x(t) - \pi(w(t))\| = 0 \\ \lim_{t\to\infty} \|\xi_m(t) - \sigma_m(w(t))\| = 0 \quad \forall\ (x(0), \xi_m(0), \xi_s(0), w(0)) \in \mathcal{D}. \\ \lim_{t\to\infty} \|\xi_s(t)\| = 0 \end{cases} \tag{16}$$

Proof. Define $\xi \triangleq [\xi_m^{\mathsf{T}}\ \xi_s^{\mathsf{T}}]^{\mathsf{T}}$ and consider the controller (9), (12), (13) which can be written in the compact form of (4) with:

$$\begin{aligned} \phi(\xi, y) &= \begin{bmatrix} \phi_m(\xi_m, y) + D_m \theta_s(\xi_s, y) + \mathrm{E}(y)\ \tilde{\psi}(\xi, y) \\ \phi_s(\xi_s, y) + \mathrm{W}(y)\ \tilde{\psi}(\xi, y) \end{bmatrix}, \\ \theta(\xi, y) &= \theta_m(\xi_m, y) + D\ \theta_s(\xi_s, y) - \tilde{\psi}(\xi, y), \end{aligned} \tag{17}$$

where $\tilde{\psi}(\xi, y) \triangleq \psi(\theta_m(\xi_m, y) + D\ \theta_s(\xi_s, y))$ and matrices $D \in \mathbb{R}^{n_u \times n_v}$ and $D_m \in \mathbb{R}^{n_m \times n_v}$ denote

$$D \triangleq \begin{bmatrix} I & 0 \end{bmatrix}, \quad D_m \triangleq \begin{bmatrix} 0 & I \end{bmatrix}. \tag{18}$$

Let $\sigma(w) = [\sigma_m^{\mathsf{T}}(w)\ \ \sigma_s^{\mathsf{T}}(w)]^{\mathsf{T}}$ be the controller zero-error steady-state mapping, where $\sigma_m(w)$ is the nonvanishing term related to the internal model states and $\sigma_s(w) = 0$ is the vanishing component related to the stabilizing controller states. Using this mapping $\sigma(w)$, the original regulation condition (6) becomes

$$\begin{cases} \begin{bmatrix} \frac{\partial \sigma_m(w)}{\partial w} s(w) \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_m(\sigma_m(w), d(w)) + \mathrm{E}(d(w))\ \tilde{\psi}(\sigma(w), d(w)) \\ \mathrm{W}(d(w))\ \tilde{\psi}(\sigma(w), d(w)) \end{bmatrix} \quad \forall\, w \in \mathcal{W}. \\ c(w) = \theta_m(\sigma_m(w), d(w)) - \tilde{\psi}(\sigma(w), d(w)) \end{cases} \tag{19}$$

From Assumption 2, it follows that the plant control input u is not saturated in the zero-error steady-state condition $u = c(w)\ \forall\, w \in \mathcal{W}$, which implies that $\tilde{\psi}(\sigma(w), d(w)) = \psi(c(w)) = 0\ \forall\, w \in \mathcal{W}$. Thus, regulation condition (6) is simplified to (14) and (15), which are independent of the anti-windup parameters. Applying now the indicated mapping $\sigma(w)$ into attraction condition (7), one obtains (16). Consequently, the conditions from Lemma 2 ensure all the ones from Lemma 1. ■

From Lemma 2, one concludes that the stabilizing controller and anti-windup design parameters of (12) and (13) do not influence the controller zero-error steady-state originally defined by (6). It follows that the controller steady-state is influenced solely by the internal model functions, as indicated in (14). Based on this reasoning, any classical internal model design approach may be employed here, as for instance, the ones discussed in Reference 22. Thus, given the prior knowledge of proper functions $\phi_m(\xi_m, y)$ and $\theta_m(\xi_m, y)$ satisfying (14) for some $\sigma_m(w)$, the sequel of the article is dedicated to the systematic design of stabilizing functions $\phi_s(\xi_s, y)$ and $\theta_s(\xi_s, y)$ and anti-windup gains E(y) and W(y) to ensure the manifold attraction conditions in (16).

A numeric nonlinear output regulation example is here presented to illustrate the proposed output regulation framework. For now, the focus is just on the solution of the regulator equations and the internal model construction. The same example case will be addressed henceforth in the article in order to demonstrate our main results related to the co-design of stabilizing and anti-windup components.

Example 1. Consider the following rational nonlinear plant and exosystem:

$$\begin{cases} \dot{x}_1 = a_1\, x_1^2\, (1+x_1^2)^{-1} + a_2\, w_2 + x_2 \\ \dot{x}_2 = a_1\, x_1\, (1+x_1^2)^{-1} + a_2\, w_1(1+w_3) + a_3\, u \end{cases}, \qquad \begin{cases} \dot{w}_1 = b_1\, (w_2 - w_1) \\ \dot{w}_2 = b_2\, w_1 - w_2 - b_4\, w_1 w_3 \\ \dot{w}_3 = b_4\, w_1 w_2 - b_3\, w_3 \end{cases}, \tag{20}$$

where $y = [x_1 \;\; w_1]^\mathsf{T}$ are the available output measurements and the target output error is defined as $e = x_1$. Terms $a_1, a_2, a_3, b_1, b_2, b_3, b_4 \in \mathbb{R}$ denote constant parameters and the control input $u \in \mathcal{U} \subset \mathbb{R}$ is bounded in a compact set $\mathcal{U} = [-\bar{u}, \bar{u}]$, where the attributed numerical value for all the aforementioned parameters are organized in Table 1.

The exosystem presented in here is the so-called Lorenz attractor, which is known for exhibiting chaotic behavior with the setup in Table 1. The Lorenz differential equations arise in a myriad of practical applications such as lasers,[24] segmented disc dynamos,[25] convection loop dynamics,[26] brushless DC motors,[27] and chemical reactions.[28] According to Li et al.,[29] it has been proven that the trajectories $w(t)$ of the Lorenz exosystem are contained in the spherical positively invariant set

$$\mathcal{W} = \{w \in \mathbb{R}^3 \,:\, \|w - w_c\| \le r\}, \quad w_c \triangleq [0 \;\; 0 \;\; b_4^{-1}(b_1 + b_2)]^\mathsf{T}, \quad r \triangleq b_3\, (b_1 + b_2) \left(2b_4\sqrt{b_3 - 1}\right)^{-1}. \tag{21}$$

The zero-error steady-state solution of the plant can be obtained by analytically solving the mappings $\pi(w) : \mathbb{R}^3 \to \mathbb{R}^2$, $c(w) : \mathbb{R}^3 \to \mathbb{R}$ and $d(w) : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the regulator Equation (5) from Lemma 1, leading to

$$\pi(w) = \begin{bmatrix} 0 \\ -a_2\, w_2 \end{bmatrix}, \quad c(w) = a_2 a_3{}^{-1}(w_2 - \tilde{b}_2\, w_1 + \tilde{b}_4\, w_1 w_3), \quad d(w) = \begin{bmatrix} 0 \\ w_1 \end{bmatrix}, \tag{22}$$

where $\tilde{b}_2 \triangleq b_2 + 1$ and $\tilde{b}_4 \triangleq b_4 - 1$. It is required next to construct proper functions $\phi_m(\xi_m, y)$ and $\theta_m(\xi_m, y)$ for the internal model stage (12) in order to satisfy the controller steady-state condition (14) in Lemma 2 for some mapping $\sigma_m(w)$. By considering traditional internal model design guidelines,[22] one can find the following solution:

$$\phi_m(\xi_m, y) = \begin{bmatrix} b_1\, (\xi_{m2} - \xi_{m1}) \\ b_2\, \xi_{m1} - \xi_{m2} - b_4\, y_2\, \xi_{m3} \\ b_4\, y_2\, \xi_{m2} - b_3\, \xi_{m3} \end{bmatrix}, \quad \theta_m(\xi_m, y) = \xi_{m2} - \tilde{b}_2\, \xi_{m1} + \tilde{b}_4\, y_2\, \xi_{m3}, \quad \sigma_m(w) = a_2 a_3^{-1}\, w, \tag{23}$$

which is inherently robust with respect to the plant parameters a_1, a_2, a_3, since they are not required to be known in order to implement $\phi_m(\xi_m, y)$ and $\theta_m(\xi_m, y)$ (i.e., the parameters are contained in the steady-state mapping $\sigma_m(w)$).

To completely solve this output regulation, one must also design the stabilizing and anti-windup components $\phi_s(\xi_s, y)$, $\theta_s(\xi_s, y)$, $E(y)$, and $W(y)$. This problem will be addressed in the sequence.

TABLE 1 Considered numerical values of all system parameters

Parameter	a_1	a_2	a_3	b_1	b_2	b_3	b_4	$\overline{u}$
Value	1/2	1	10^3	10	28	8/3	1	4

Note: The exosystem setup is as in Reference 29.

3.2 | Stabilization problem setup using the DAR

To systematically design a stabilizing controller in the form of (13), as considered in Reference 21, we first introduce an auxiliary proxy error signal

$$\varepsilon = \delta(y), \tag{24}$$

which is defined by a steady-state vanishing function $\delta : \mathbb{R}^{n_y} \to \mathbb{R}^{n_\varepsilon}$ of the available measurements y, that is

$$0 = \delta(d(w)) \ \ \forall\, w \in \mathcal{W}. \tag{25}$$

This proxy error ε represents a steady-state vanishing signal to be employed as a feedback component in the stabilizing controller. Note that whenever the original output error e is implementable with the measurements, that is, $\exists\ \mathrm{h}(y)$: $\mathrm{h}(g(x, w)) = h(x, w)$, then $\delta(y) = \mathrm{h}(y)$ can be considered, in which case ε is equivalent to e. In the case from Example 1, for instance, the proxy error can be chosen as $\varepsilon = \delta(y) = y_1$. The methodology in this article, however, will be also capable of dealing with cases where ε is different than e, provided that (25) holds for a given $\delta(y)$. Based on this proxy error signal ε, the stabilizing controller (13) is particularly considered in the following form:

$$\begin{cases} \dot{\xi}_s = \mathrm{F}(y)\ \xi_s + \mathrm{G}(y)\ \varepsilon + \mathrm{W}(y)\ \psi(\mu) \\ v = \mathrm{H}(y)\ \xi_s + \mathrm{K}(y)\ \varepsilon \end{cases}, \tag{26}$$

where $\mathrm{F} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_s}$, $\mathrm{G} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_\varepsilon}$, $\mathrm{H} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_v \times n_s}$ and $\mathrm{K} : \mathbb{R}^{n_y} \to \mathbb{R}^{n_v \times n_\varepsilon}$ are free-design stabilizing matrices possibly dependent on the available measurements. From the vanishing condition (25), it is readily noticeable that this controller structure satisfies the steady-state requirement (15).

Thus, considering a priori known internal model functions $\phi_m(\xi_m, y)$, and $\theta_m(\xi_m, y)$ satisfying (14) for some $\sigma_m(w)$ and a function $\delta(y)$ satisfying (25), one should design $\mathrm{F}(y)$, $\mathrm{G}(y)$, $\mathrm{H}(y)$, $\mathrm{K}(y)$, $\mathrm{E}(y)$, and $\mathrm{W}(y)$ such that the attraction requirement (16) is satisfied in some region $\mathcal{D} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_s} \times \mathcal{W}$. With this aim, let us introduce a regulation error state $z \in \mathbb{R}^{n_z}$ ($n_z = n_x + n_m$) defined as follows:

$$z \triangleq \begin{bmatrix} z_x \\ z_m \end{bmatrix} \triangleq \begin{bmatrix} x - \pi(w) \\ \xi_m - \sigma_m(w) \end{bmatrix}. \tag{27}$$

The vector component $z_x \in \mathbb{R}^{n_x}$ here denotes the deviation between the system state x and the regulation reference $\pi(w)$, while $z_m \in \mathbb{R}^{n_m}$ denotes the deviation between the internal model state ξ_m and its regulation reference $\sigma_m(w)$. Noting that $z \to 0$ implies that $x \to \pi(w)$ and $\xi_m \to \sigma_m(w)$, the manifold stabilization problem related to condition (16) may be regarded by the asymptotic stabilization of the trajectories $z(t)$ with respect to the origin. Since it is also necessary to ensure that $\xi_s \to 0$, that is, the asymptotic convergence of the stabilizing controller states, it is useful to define an augmented regulation error state as

$$\mathbf{z} \triangleq \begin{bmatrix} z \\ \xi_s \end{bmatrix}. \tag{28}$$

Now, the target attraction condition from (16) can be rewritten in the following compact way:

$$\lim_{t\to\infty} \|\mathbf{z}(t)\| = 0 \ \ \forall\ (\mathbf{z}(0), w(0)) \in \mathcal{E} \subseteq \mathbb{R}^{n_a} \times \mathcal{W}, \tag{29}$$

where $\mathcal{E}$ denotes region $\mathcal{D}$ remapped from the original (x, ξ_m, ξ_s, w) space to the regulation error $(\mathbf{z}, w)$ space. Thus, the output regulation problem has been fully reframed as an origin asymptotic stabilization problem. In the sequence, we focus on the dynamical description of the variables z and ξ_s in order to completely setup this stabilization problem.

Let us now consider an effective stabilizing input signal $\tilde{\nu} \in \mathbb{R}^{n_\nu}$ given by:

$$\tilde{\nu} \triangleq \begin{bmatrix} \tilde{\nu}_u \\ \tilde{\nu}_m \end{bmatrix} \triangleq \begin{bmatrix} \nu_u - \psi(\mu) \\ \nu_m + \mathrm{E}(y)\ \psi(\mu) \end{bmatrix}. \tag{30}$$

By using the matrices defined in (18), one should note that (30) can be equivalently written as

$$\tilde{\nu} = \nu + (D_m^\mathsf{T}\mathrm{E}(y) - D^\mathsf{T})\ \psi(\mu). \tag{31}$$

The dynamics of the state z and other important variables such as the unconstrained control signal μ and the proxy error signal ε may then be expressed in the following manner, which will be referred as the regulation error system:

$$\begin{cases} \dot{z} = f_z(z, w, \tilde{\nu}) \\ \mu = \theta_z(z, w) + c(w) + \nu_u \\ \varepsilon = \delta_z(z, w) \end{cases}, \tag{32}$$

where these new system functions are obtained substituting x by $z_x + \pi(w)$ and ξ_m by $z_m + \sigma_m(w)$ in the original system Equations (1) and (12), which yields:

$$\begin{aligned} f_z(z, w, \tilde{\nu}) &= \begin{bmatrix} f(z_x + \pi(w), w, \theta_z(z, w) + c(w) + \tilde{\nu}_u) \\ \phi_m(z_m + \sigma_m(w), g_z(z, w) + d(w)) + \tilde{\nu}_m \end{bmatrix} - \begin{bmatrix} f(\pi(w), w, c(w)) \\ \phi_m(\sigma_m(w), d(w)) \end{bmatrix}, \\ \theta_z(z, w) &= \theta_m(z_m + \sigma_m(w), g_z(z, w) + d(w)) - c(w), \\ \delta_z(z, w) &= \delta(g_z(z, w) + d(w)), \end{aligned} \tag{33}$$

where $g_z(z, w) + d(w) \triangleq g(z_x + \pi(w), w) - d(w)$ is an auxiliary definition. From the previously established steady-state conditions, an important characteristic of the representation in (32) is the fact that relations $f_z(0, w, 0) = 0$, $\theta_z(0, w) = 0$, $\delta_z(0, w) = 0$, and $g_z(0, w) = 0$ are verified $\forall\ w \in \mathcal{W}$.

In order to deal with the nonlinear functions of the regulation error system (32), we consider a DAR. This procedure is applicable whenever these functions are regionally regular rational[30] with respect to variables (z, w) inside some validity domain $\mathcal{Z}^+ \times \mathcal{W}^+$. One should note that set $\mathcal{W}^+$ must at least contain $\mathcal{W}$, where the exosystem trajectories are bounded from Assumption 1, whereas set $\mathcal{Z}^+$ must at least contain the origin $z = 0$, which is the target equilibrium point. To provide a numerically tractable method, we henceforth regard these sets $\mathcal{Z}^+$ and $\mathcal{W}^+$ as a priori defined convex polytopes, which can be described by a convex hull of its vertices, with $\mathcal{V}\{\mathcal{Z}^+\}$ and $\mathcal{V}\{\mathcal{W}^+\}$ denoting the set of vertices of $\mathcal{Z}^+$ and $\mathcal{W}^+$, respectively. Without any loss of generality, we also consider the set $\mathcal{Z}^+$ expressed in the following alternative form:

$$\mathcal{Z}^+ = \left\{ z \in \mathbb{R}^{n_z} : |p_k^\mathsf{T} z| \leq 1,\ k = 1, 2, \ldots, n_k \right\}, \tag{34}$$

with $p_1, p_2, \ldots, p_{n_k} \in \mathbb{R}^{n_z}$.

Remark 1. In Assumption 3, we assumed that all steady-state mappings and system functions are regionally regular rational in order to employ the DAR framework. Note that this assumption can be relaxed, since in fact only the resultant composed functions $f_z(z, w, \tilde{\nu})$, $\theta_z(z, w)$, $\delta_z(z, w)$ and the mapping $c(w)$, which appear in (32), are required to be regionally regular rational inside $\mathcal{Z}^+ \times \mathcal{W}^+$.

According to Coutinho et al.,[30] it follows from Assumption 3 that a proper well-posed DAR can always be chosen in order to deal with the rational functions $f_z(z, w, \tilde{\nu})$, $\theta_z(z, w)$, and $\delta_z(z, w)$ in system (32). For instance, in this article we consider:

$$f_z(z,w,\tilde{v}) = A(z,w)\ z + \Phi(z,w)\ \varphi(z,w) + B\ \tilde{v}$$
$$\theta_z(z,w) = Q(z,w)\ z + \Upsilon(z,w)\ \varphi(z,w) \tag{35}$$
$$\delta_z(z,w) = C\ z + \Gamma\ \varphi(z,w)$$

with a rational nonlinear mapping $\varphi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi}$ satisfying

$$0 \ = \ \Psi(z,w)\ z + \Omega(z,w)\ \varphi(z,w). \tag{36}$$

Matrices $A : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_z \times n_z}$, $\Phi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_z \times n_\varphi}$, $Q : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_u \times n_z}$, $\Upsilon : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_u \times n_\varphi}$, $\Psi : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi \times n_z}$, and $\Omega : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi \times n_\varphi}$ are affine with respect to (z,w) and matrices $B \in \mathbb{R}^{n_z \times n_v}$, $C \in \mathbb{R}^{n_\varepsilon \times n_z}$ and $\Gamma \in \mathbb{R}^{n_\varepsilon \times n_\varphi}$ are constant[2]. It is considered that matrix $\Omega(z,w)$ is nonsingular $\forall\ (z,w) \in \mathcal{Z}^+ \times \mathcal{W}^+$, which ensures the well-posedness of the DAR.

Since the system matrix $A(z,w)$ was considered affine in (z,w), one can always express it in the form of

$$A(z,w) = \mathrm{A}_0 + \sum_{i=1}^{n} \mathrm{A}_i\ \lambda_i(z,w), \tag{37}$$

where $\mathrm{A}_0, \ldots, \mathrm{A}_\mathrm{n} \in \mathbb{R}^{n_z \times n_z}$ are constant matrices and $\lambda : \mathbb{R}^{n_z} \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is a linear mapping. The role of the latter is to serve as a gain-scheduling vector function, with the purpose of providing additional degree of freedom to the stabilization problem. The restriction on the choice of $\lambda(z,w)$, besides the linearity, is its implementability with the available measurement y, that is

$$\exists\ \lambda(y) : \lambda(g_z(z,w) + d(w)) = \lambda(z,w). \tag{38}$$

Given these considerations, we define the stabilizing controller gains $\mathrm{F}(y)$, $\mathrm{G}(y)$, $\mathrm{H}(y)$, $\mathrm{K}(y)$ and the anti-windup gains $\mathrm{E}(y)$, $\mathrm{W}(y)$ such as the following affine parametrization:

$$\begin{bmatrix} \mathrm{F}(y) & \mathrm{G}(y) \\ \mathrm{H}(y) & \mathrm{K}(y) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{F}_0 & \mathrm{G}_0 \\ \mathrm{H}_0 & \mathrm{K}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \mathrm{F}_i & \mathrm{G}_i \\ \mathrm{H}_i & \mathrm{K}_i \end{bmatrix} \lambda_i(y)\ , \quad \begin{bmatrix} \mathrm{E}(y) \\ \mathrm{W}(y) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{E}_0 \\ \mathrm{W}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \mathrm{E}_i \\ \mathrm{W}_i \end{bmatrix} \lambda_i(y), \tag{39}$$

where $\mathrm{F}_i,\ \mathrm{G}_i,\ \mathrm{H}_i,\ \mathrm{K}_i,\ \mathrm{E}_i,\ \mathrm{W}_i\ \forall i \in \{0, \ldots, n\}$ are free constant matrices to be designed. From relation (38), it follows that (39) can be equivalently expressed in terms of (z,w) according to

$$\begin{bmatrix} F(z,w) & G(z,w) \\ H(z,w) & K(z,w) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{F}_0 & \mathrm{G}_0 \\ \mathrm{H}_0 & \mathrm{K}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \mathrm{F}_i & \mathrm{G}_i \\ \mathrm{H}_i & \mathrm{K}_i \end{bmatrix} \lambda_i(z,w)\ , \quad \begin{bmatrix} E(z,w) \\ W(z,w) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{E}_0 \\ \mathrm{W}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \mathrm{E}_i \\ \mathrm{W}_i \end{bmatrix} \lambda_i(z,w). \tag{40}$$

Whereas Equation (39) reflects the actual implementation of the controller and anti-windup gains, the alternative notation from (40) will be useful to derive our proposed stabilization framework in the sequence.

Considering that $f_z(z,w,\tilde{v})$, $\theta_z(z,w)$, and $\delta_z(z,w)$ are equivalent to (35) and (36) and applying the deadzone effect from (30), we can now write the regulation error system (32) as follows:

$$\begin{cases} \dot{z} = A(z,w)\ z + \Phi(z,w)\ \varphi(z,w) + B\ (D_m^\mathsf{T} E(z,w) - D^\mathsf{T})\ \psi(\mu) + B\ v \\ \mu = Q(z,w)\ z + \Upsilon(z,w)\ \varphi(z,w) + c(w) + D\ v \\ \varepsilon = C\ z + \Gamma\ \varphi(z,w) \\ 0 = \Psi(z,w)\ z + \Omega(z,w)\ \varphi(z,w) \end{cases} . \tag{41}$$

[2]There is no loss of generality in considering these matrices as constant, since the nonlinearities can be lumped into $\varphi(z,w)$. Moreover, this consideration will ensure that all closed-loop system matrices will remain affine in (z,w), a desirable property for simplifying the control synthesis.

Then, using the augmented state vector $\mathbf{z}$ defined in (28), we can express the complete closed-loop dynamics, which combines the regulation error system (41) with the proposed stabilizing stage (26):

$$\begin{cases} \dot{\mathbf{z}} = \boldsymbol{A}(z,w)\ \mathbf{z} + \boldsymbol{\Phi}(z,w)\ \varphi(z,w) + \boldsymbol{J}(z,w)\ \psi(\mu) \\ \mu = \boldsymbol{Q}(z,w)\ \mathbf{z} + \boldsymbol{\Upsilon}(z,w)\ \varphi(z,w) + c(w) \\ 0 = \boldsymbol{\Psi}(z,w)\ \mathbf{z} + \boldsymbol{\Omega}(z,w)\ \varphi(z,w) \end{cases}, \tag{42}$$

where matrices $\boldsymbol{A} : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_a \times n_a}$, $\boldsymbol{\Phi} : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_a \times n_\varphi}$, $\dots$, $\boldsymbol{\Omega} : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_\varphi \times n_\varphi}$ are given as follows:

$$\begin{aligned}
\boldsymbol{A}(z,w) &= \begin{bmatrix} A(z,w) + BK(z,w)C & BH(z,w) \\ G(z,w)C & F(z,w) \end{bmatrix}, \\
\boldsymbol{\Phi}(z,w) &= \begin{bmatrix} \Phi(z,w) + BK(z,w)\Gamma \\ G(z,w)\Gamma \end{bmatrix}, \\
\boldsymbol{J}(z,w) &= \begin{bmatrix} BD_m^{\mathsf{T}}E(z,w) - BD^{\mathsf{T}} \\ W(z,w) \end{bmatrix}, \\
\boldsymbol{Q}(z,w) &= \begin{bmatrix} Q(z,w) + DK(z,w)C & DH(z,w) \end{bmatrix}, \\
\boldsymbol{\Upsilon}(z,w) &= \Upsilon(z,w) + DK(z,w)\Gamma\ , \\
\boldsymbol{\Psi}(z,w) &= \begin{bmatrix} \Psi(z,w) & 0 \end{bmatrix}, \\
\boldsymbol{\Omega}(z,w) &= \Omega(z,w)\ .
\end{aligned} \tag{43}$$

The solution of Problem 1 can therefore be achieved by the determination of matrices F_i, G_i, H_i, K_i, E_i, W_i $\forall i \in \{0, \dots, n\}$ such that the trajectories of system (42) converge to the origin as in (29) for some region of initial conditions $\mathcal{E} \subseteq \mathbb{R}^{n_a} \times \mathcal{W}$. A systematic procedure to that will be detailed in Section 3.4.

In order to illustrate the setup described previously, in the next example we show how to construct the regulation error dynamics and its DAR for the same system considered in Example 1.

Example 2. Consider the system and exosystem from (20) with steady-state mappings and internal model functions designed as in Example 1. We first choose $\varepsilon = \delta(y) = y_1$, noting that (25) is satisfied since $d_1(w) = 0$. Second, we express the system using the regulation error coordinate change introduced by (27), which gives

$$z_1 \triangleq x_1, \quad z_2 \triangleq x_2 + a_2\ w_2, \quad z_3 \triangleq \xi_{m1} - a_2 a_3^{-1}\ w_1, \quad z_4 \triangleq \xi_{m2} - a_2 a_3^{-1}\ w_2, \quad z_5 \triangleq \xi_{m3} - a_2 a_3^{-1}, w_3. \tag{44}$$

Thus, functions $\dot{z} = f_z(z, w, \tilde{v})$, $\mu = \theta(z,w)$, and $\varepsilon = \delta_z(z,w)$, from the regulation error system, are given by:

$$f_z(z,w,\tilde{v}) = \begin{bmatrix} a_1\ z_1^2\ (1+z_1^2)^{-1} + z_2 \\ a_1\ z_1\ (1+z_1^2)^{-1} + a_3\ (z_4 - \tilde{b}_2\ z_3 + \tilde{b}_4\ w_1 z_5 + \tilde{v}_1) \\ b_1\ (z_4 - z_3) + \tilde{v}_2 \\ b_2\ z_3 - z_4 - b_4\ w_1 z_5 + \tilde{v}_3 \\ b_4\ w_1 z_4 - b_3\ z_5 + \tilde{v}_4 \end{bmatrix}, \qquad \begin{aligned} \theta_z(z,w) &= z_4 - \tilde{b}_2\ z_3 + \tilde{b}_4\ w_1 z_5, \\ \delta_z(z) &= z_1. \end{aligned} \tag{45}$$

Given this representation, the next step is to decompose $f_z(z, w, \tilde{v})$, $\theta_z(z,w)$, $\delta_z(z)$ into an appropriate DAR as stated by Equations (35) and (36). For such procedure, one can choose the following vector of rational nonlinearities:

$$\varphi(z) = \begin{bmatrix} z_1^2\ (1+z_1^2)^{-1} \\ z_1\ (1+z_1^2)^{-1} \end{bmatrix}. \tag{46}$$

Matrices in (35) and (36) are then specified as follows:

$$A(w)=\begin{bmatrix}0&1&0&0&0\\0&0&-a_3\tilde{b}_2&a_3&a_3\tilde{b}_4w_1\\0&0&-b_1&b_1&0\\0&0&b_2&-1&-b_4w_1\\0&0&0&b_4w_1&-b_3\end{bmatrix},\quad \Phi=\begin{bmatrix}a_1&0\\0&a_1\\0&0\\0&0\\0&0\end{bmatrix},\quad B=\begin{bmatrix}0&0&0&0\\a_3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$
$$Q(w)=\begin{bmatrix}0&0&-\tilde{b}_2&1&\tilde{b}_4\,w_1\end{bmatrix},\qquad \Upsilon=\begin{bmatrix}0&0\end{bmatrix},$$
$$C=\begin{bmatrix}1&0&0&0&0\end{bmatrix},\qquad \Gamma=\begin{bmatrix}0&0\end{bmatrix},$$
$$\Psi=\begin{bmatrix}0&0&0&0&0\\-1&0&0&0&0\end{bmatrix},\qquad \Omega(z)=\begin{bmatrix}1&-z_1\\z_1&1\end{bmatrix},\tag{47}$$

where $\Omega(z)$ is nonsingular $\forall\ z\in\mathbb{R}^5$ because $\det\{\Omega(z)\}=1+z_1^2\geq 1$. Note that the system matrix $A(w)$ is in this case an affine function of $\lambda(w)=w_1$, that is, $A(w)=\mathrm{A}_0+\mathrm{A}_1w_1$, with:

$$\mathrm{A}_0=\begin{bmatrix}0&1&0&0&0\\0&0&-a_3\tilde{b}_2&a_3&0\\0&0&-b_1&b_1&0\\0&0&b_2&-1&0\\0&0&0&0&-b_3\end{bmatrix},\quad \mathrm{A}_1=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&a_3\tilde{b}_4\\0&0&0&0&0\\0&0&0&0&-b_4\\0&0&0&b_4&0\end{bmatrix}.\tag{48}$$

According to the proposed setup procedure, the signal $\lambda(w)=w_1$, which is equivalent to $\lambda(y)=y_2$, will also be utilized as a gain-scheduling variable for the stabilizing controller and anti-windup gains.

3.3 | Proposed sector condition

The purpose of this subsection is to present a new modified sector condition associated with the deadzone function $\psi(\mu)$ which appears in (42) as a consequence of the possible control signal saturation. This result will be essential to derive conditions in the form of matrix inequalities to solve the stabilization problem.

Differently from the modified sector condition proposed in Reference 23, which was developed for a standard stabilization problem, our new condition is able to account for the nonvanishing characteristic of μ toward the steady-state equilibrium point $\mathbf{z}=0$, which is a particular characteristic of the output regulation problem. For instance, our new condition allows one to address the presence of $\psi(\mu)$ into system by just considering the vanishing component $\boldsymbol{\theta}(\mathbf{z},w)\triangleq\boldsymbol{Q}(z,w)\,\mathbf{z}+\boldsymbol{\Upsilon}(z,w)\,\varphi(z,w)$ of the signal μ and the magnitude upper-bounds of its nonvanishing component $c(w)$. This result is stated as follows.

Lemma 3. *Consider functions $\boldsymbol{\theta},\boldsymbol{\vartheta}:\mathbb{R}^{n_a}\times\mathbb{R}^{n_w}\to\mathbb{R}^{n_u}$ and positive scalars $\bar{c}_1,\ldots,\bar{c}_{n_u}\in\mathbb{R}$ such that*

$$\sup_{w\in\mathcal{W}}|c_j(w)|\leq\bar{c}_j<\bar{u}_j\quad\forall\,j\in\{1,2,\ldots,n_u\}.\tag{49}$$

If $(\mathbf{z},w)\in\mathcal{S}$, where $\mathcal{S}$ is the set

$$\mathcal{S}=\{(\mathbf{z},w)\in\mathbb{R}^{n_a}\times\mathcal{W}:\ |\theta_j(\mathbf{z},w)-\vartheta_j(\mathbf{z},w)\,|\ \leq(\bar{u}_j-\bar{c}_j),\ j=1,2,\ldots,n_u\},\tag{50}$$

then it follows that

$$\psi^{\mathsf{T}}(\mu)\ T\ (\psi(\mu)-\boldsymbol{\vartheta}(\mathbf{z},w))\leq 0\tag{51}$$

is verified for $\mu=\boldsymbol{\theta}(\mathbf{z},w)+c(w)$ and any diagonal matrix $T\in\mathbb{R}^{n_u\times n_u}$, $T>0$.

Proof. Observe all the possible cases that follow $\forall\, j = 1, 2, \dots, n_u$ when $\psi(\mu_j) \neq 0$:

(a) $\psi(\mu_j) > 0$. In this case it follows that $\psi(\mu_j) = \theta_j(\mathbf{z}, w) + c_j(w) - \overline{u}_j \leq \theta_j(\mathbf{z}, w) + \overline{c}_j - \overline{u}_j$, where $\overline{c}_j$ is as in (49). Moreover, it is also verified that $\psi(\mu_j) - \vartheta_j(\mathbf{z}, w) \leq \theta_j(\mathbf{z}, w) - \vartheta_j(\mathbf{z}, w) + \overline{c}_j - \overline{u}_j$. Furthermore, if $(\mathbf{z}, w) \in \mathcal{S}$, as defined by (50), it follows that $\theta_j(\mathbf{z}, w) - \vartheta_j(\mathbf{z}, w) \leq \overline{u}_j - \overline{c}_j$ which implies that $\psi(\mu_j) - \vartheta_j(\mathbf{z}, w) \leq \theta_j(\mathbf{z}, w) + \overline{c}_j - \overline{u}_j - \vartheta_j(\mathbf{z}, w) \leq 0$. Thus, as $\psi(\mu_j) > 0$, we conclude that:

$$\psi^{\mathsf{T}}(\mu_j)\ T_{[j,j]}\ (\psi(\mu_j) - \vartheta_j(\mathbf{z}, w)) \leq 0. \tag{52}$$

(b) $\psi(\mu_j) < 0$. In this case it follows that $\psi(\mu_j) = \theta_j(\mathbf{z}, w) + c_j(w) + \overline{u}_j \geq \theta_j(\mathbf{z}, w) - \overline{c}_j + \overline{u}_j$ and also $\psi(\mu_j) - \vartheta_j(\mathbf{z}, w) \geq \theta_j(\mathbf{z}, w) - \vartheta_j(\mathbf{z}, w) - \overline{c}_j + \overline{u}_j$. Furthermore, if $(\mathbf{z}, w) \in \mathcal{S}$, it follows that $\theta_j(\mathbf{z}, w) - \vartheta_j(\mathbf{z}, w) \geq -(\overline{u}_j - \overline{c}_j)$ which implies that $\psi(\mu_j) - \vartheta_j(\mathbf{z}, w) \geq \theta_j(\mathbf{z}, w) - \vartheta_j(\mathbf{z}, w) - \overline{c}_j + \overline{u}_j \geq 0$. Thus, as $\psi(\mu_j) < 0$, we conclude that (52) is again verified.

From all these possible cases, as $\psi(\mu)$ is a decentralized nonlinearity, it follows that provided (49) holds and $(\mathbf{z}, w) \in \mathcal{S}$, relation (51) is verified for any diagonal matrix $T \succ 0$. ■

To complement the result provided by Lemma 3, we have also devised a method to solve the problem of finding upper-bound parameters $\overline{c}_1, \dots, \overline{c}_{n_u}$ in (49) by semidefinite programming. This approach is applicable whenever $c(w)$ is a regular rational mapping with respect to $\mathcal{W}^+ \supseteq \mathcal{W}$. With this assumption, one can always express the function $c(w)$ in a proper well-posed algebraic representation, such as

$$c(w) = \tilde{Q}(w)\ w + \tilde{\Upsilon}(w)\ \eta(w) \tag{53}$$

with a rational nonlinear mapping $\eta : \mathcal{W}^+ \to \mathbb{R}^{n_\eta}$ satisfying

$$0 = \tilde{\Psi}(w)\ w + \tilde{\Omega}(w)\ \eta(w). \tag{54}$$

Matrices $\tilde{Q} : \mathcal{W}^+ \to \mathbb{R}^{n_u \times n_w}$, $\tilde{\Upsilon} : \mathcal{W}^+ \to \mathbb{R}^{n_u \times n_\eta}$, $\tilde{\Psi} : \mathcal{W}^+ \to \mathbb{R}^{n_\eta \times n_w}$ and $\tilde{\Omega} : \mathcal{W}^+ \to \mathbb{R}^{n_\eta \times n_\eta}$ are affine with respect to w and $\tilde{\Omega}(w)$ is nonsingular $\forall\, w \in \mathcal{W}^+$. Also without loss of generality, we introduce an auxiliary ellipsoidal bound in the form of

$$\tilde{\mathcal{W}} = \left\{ w \in \mathbb{R}^{n_w} : (w - w_c)^{\mathsf{T}} \tilde{P}\ (w - w_c) \leq 1 \right\}, \tag{55}$$

for a symmetric and positive definite matrix $\tilde{P} \in \mathbb{R}^{n_w \times n_w}$ and a center point $w_c \in \mathbb{R}^{n_w}$ such that $\mathcal{W} \subseteq \tilde{\mathcal{W}} \subseteq \mathcal{W}^+$. Given these considerations, we obtained the following result.

Lemma 4. *If there exist matrices $\tilde{L}_j \in \mathbb{R}^{n_\eta \times n_\eta}$ and scalars $\gamma_j \in \mathbb{R}\ \forall\, j \in \{1, 2, \dots, n_u\}$ such that:*

$$\gamma_j > 0, \quad \begin{bmatrix} \gamma_j & \tilde{Q}_{[j]}(w) & \tilde{\Upsilon}_{[j]}(w) & 0 \\ \star & \tilde{P} & \tilde{\Psi}^{\mathsf{T}}(w)\tilde{L}_j^{\mathsf{T}} & \tilde{P}w_c \\ \star & \star & \mathrm{He}\{\tilde{L}_j\tilde{\Omega}(w)\} & 0 \\ \star & \star & \star & w_c^{\mathsf{T}}\tilde{P}w_c \end{bmatrix} \succcurlyeq 0\ \forall\, w \in \mathcal{V}\{\mathcal{W}^+\}, \tag{56}$$

then (49) is satisfied with $\overline{c}_j = \sqrt{\gamma_j}$.

Proof. Suppose that (56) is verified. From Schür's complement and from the fact that the left-hand side of (56) is affine in w, it follows that

$$\begin{bmatrix} w^{\mathsf{T}} & \eta^{\mathsf{T}}(w) & -1 \end{bmatrix} \left(\begin{bmatrix} \tilde{P} & \tilde{\Psi}^{\mathsf{T}}(w)\tilde{L}_j^{\mathsf{T}} & \tilde{P}w_c \\ \star & \mathrm{He}\{\tilde{L}_j\tilde{\Omega}(w)\} & 0 \\ \star & \star & w_c^{\mathsf{T}}\tilde{P}w_c \end{bmatrix} - \begin{bmatrix} \tilde{Q}_{[j]}^{\mathsf{T}}(w) \\ \tilde{\Upsilon}_{[j]}^{\mathsf{T}}(w) \\ 0 \end{bmatrix} \frac{1}{\gamma_j} \begin{bmatrix} \tilde{Q}_{[j]}(w) & \tilde{\Upsilon}_{[j]}(w) & 0 \end{bmatrix} \right) \begin{bmatrix} w \\ \eta(w) \\ -1 \end{bmatrix} \geq 0\ \forall\, w \in \mathcal{W}^+ . \tag{57}$$

From (53) and (54), we then have that $\tilde{\Psi}(w)\,w + \tilde{\Omega}(w)\,\eta(w) = 0$ and $\tilde{Q}_{[j]}(w)\,w + \tilde{\Upsilon}_{[j]}(w)\,\eta(w) = c_j(w)$, so (57) reduces to:

$$(w - w_c)^{\mathsf{T}} \tilde{P}\,(w - w_c) - c_j^{\mathsf{T}}(w)\,\gamma_j^{-1} c_j(w) \geq 0 \;\forall\, w \in \mathcal{W}^+ \quad \Leftrightarrow \quad \gamma_j^{-1} |c_j(w)|^2 \leq (w - w_c)^{\mathsf{T}} \tilde{P}\,(w - w_c) \;\forall\, w \in \mathcal{W}^+. \tag{58}$$

Consider now the set $\tilde{\mathcal{W}} \subseteq \mathcal{W}^+$ as defined in (55). Hence, from (58), we conclude that

$$|c_j(w)|^2 \leq \gamma_j \;\forall\, w \in \tilde{\mathcal{W}}. \tag{59}$$

Moreover, since $\tilde{\mathcal{W}}$ is defined such that $\mathcal{W} \subseteq \tilde{\mathcal{W}}$, relation (60) also implies that

$$|c_j(w)|^2 \leq \gamma_j \;\forall\, w \in \mathcal{W}, \tag{60}$$

which finally leads to:

$$\sup_{w \in \mathcal{W}} |c_j(w)| \leq \sqrt{\gamma_j}. \tag{61}$$

Thus, (49) holds with $\bar{c}_j = \sqrt{\gamma_j}$ if conditions in (56) are verified. ∎

Based on Lemma 4, each steady-state control bound $\bar{c}_j$ can be determined by the following semidefinite optimization:

$$\min_{\tilde{L}_j,\, \gamma_j} \;\gamma_j \quad \text{s.t.} \quad (56). \tag{62}$$

The subsequent example illustrates the employment of this method in a numerical case study.

Example 3. Consider the system and exosystem from Example 1, where the zero-error steady-state control mapping was verified to be $c(w) = a_2 a_3^{-1}(w_2 - \tilde{b}_2\, w_1 + \tilde{b}_4\, w_1 w_3)$ with $\tilde{b}_2 \triangleq b_2 + 1$ and $\tilde{b}_4 \triangleq b_4 - 1$. As also mentioned in Example 1, the exosystem trajectories $w(t)$ are contained inside a spherical positively invariant set with radius r and center w_c, as defined in (21).

We apply now Lemma 4 to find a lowest upper-bound $\bar{c}$ for $c(w)\;\forall\, w \in \mathcal{W}$. For this, it is first necessary to express $c(w)$ as defined in (53) and (54). A possible representation in this case is obtained with

$$\tilde{Q}(w) = a_2 a_3^{-1} \begin{bmatrix} -\tilde{b}_2 & 1 & 0 \end{bmatrix}, \quad \tilde{\Upsilon} = \tilde{b}_4\,, \quad \tilde{\Psi}(w) = \begin{bmatrix} 0 & 0 & -w_1 \end{bmatrix}, \quad \tilde{\Omega} = 1, \tag{63}$$

where $\eta(w) = w_1 w_3$. The ellipsoidal set $\tilde{\mathcal{W}}$ can be defined with $\tilde{P} = I r^{-2}$ and with the same previously mentioned center point w_c, noting that $\mathcal{W} = \tilde{\mathcal{W}}$ in this case. Moreover, since the system matrices only depend on w_1, set $\mathcal{W}^+$ does not need to be bounded in the directions associated to w_2 and w_3. For instance, one may consider $\mathcal{W}^+ = [-r,\ r] \times \mathbb{R}^2$, which implies that $\mathcal{W} = \tilde{\mathcal{W}} \subset \mathcal{W}^+$.

Considering the numerical parameters from Table 1, we determined that $\bar{c} = 1.1388$ by solving the semidefinite programming (62). The knowledge of this upper-bound particularly allows the employment of the sector condition from Lemma 3 for control design purposes, as it will be seen in Section 3.4.

3.4 | Main results: Stability and performance conditions

This section presents our main results regarding the development of the closed-loop stability conditions in the form of matrix inequalities. These conditions are subsequently used to cast the co-design of stabilizing controller and anti-windup parameters by numerical optimization methods.

Beyond dealing with the original asymptotic output regulation requirement of the closed-loop system, which was emphasized by the requirement (29), we will also consider an exponential convergence criterion. With this aim, consider the following definition.

Definition 3. α-Exponential Convergence: The augmented regulation error state trajectories $\mathbf{z}(t)$ are said to converge exponentially to zero in $\mathcal{E}$ if there exist scalars $\beta > 0$ and $\alpha > 0$ such that

$$\|\mathbf{z}(t)\| \leq \beta \, \|\mathbf{z}(0)\| \, \mathrm{e}^{-\alpha t} \; \forall \; t \geq 0, \; \forall \; (\mathbf{z}(0), w(0)) \in \mathcal{E} \subseteq \mathbb{R}^{n_a} \times \mathcal{W}. \tag{64}$$

In this case, the trajectories $\mathbf{z}(t)$ approach the origin with an exponential decay rate greater then α.

Based on the previous definitions and the proposed sector condition from Lemma 3, we can now formulate conditions to guarantee the α-exponential convergence of the regulation error system trajectories to the origin, provided that the initial conditions $(\mathbf{z}(0), w(0))$ belong to a certain region $\mathcal{E}$.

Theorem 1. *Assume $n_s = n_z$. Suppose that there exist symmetric matrices $X, Y \in \mathbb{R}^{n_z \times n_z}$, a diagonal matrix $\hat{T} \in \mathbb{R}^{n_u \times n_u}$ and matrices $L \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $\hat{\mathrm{R}}_0, \dots, \hat{\mathrm{R}}_m \in \mathbb{R}^{n_u \times n_z}$, $\hat{\Xi}_0, \dots, \hat{\Xi}_m \in \mathbb{R}^{n_u \times n_s}$, $\hat{\Pi}_0, \dots, \hat{\Pi}_m \in \mathbb{R}^{n_u \times n_\varphi}$, $\hat{\mathrm{F}}_0, \dots, \hat{\mathrm{F}}_\mathrm{n} \in \mathbb{R}^{n_z \times n_z}$, $\hat{\mathrm{G}}_0, \dots, \hat{\mathrm{G}}_\mathrm{n} \in \mathbb{R}^{n_z \times n_\epsilon}$, $\hat{\mathrm{H}}_0, \dots, \hat{\mathrm{H}}_\mathrm{n} \in \mathbb{R}^{n_v \times n_z}$, $\hat{\mathrm{K}}_0, \dots, \hat{\mathrm{K}}_\mathrm{n} \in \mathbb{R}^{n_v \times n_\epsilon}$, $\hat{\mathrm{E}}_0, \dots, \hat{\mathrm{E}}_\mathrm{n} \in \mathbb{R}^{n_m \times n_u}$, and $\hat{\mathrm{W}}_0, \dots, \hat{\mathrm{W}}_\mathrm{n} \in \mathbb{R}^{n_z \times n_u}$ such that the matrix inequalities:*

$$\hat{T} \succ 0, \quad \begin{bmatrix} X & I \\ \star & Y \end{bmatrix} \succ 0, \quad \begin{bmatrix} 1 & p_k^\mathsf{T} X & p_k^\mathsf{T} \\ \star & X & I \\ \star & \star & Y \end{bmatrix} \succ 0 \; \forall \; k \in \{1, \dots, n_k\}, \tag{65}$$

$$\begin{bmatrix} (\overline{u}_j - \overline{c}_j)^2 & M_{1[j]}(z,w) & M_{2[j]}(z,w) & M_{3[j]}(z,w) \\ \star & X & I & -X\Psi^\mathsf{T}(z,w)L^\mathsf{T} \\ \star & \star & Y & -\Psi^\mathsf{T}(z,w)L^\mathsf{T} \\ \star & \star & \star & -\mathrm{He}\{L\Omega(z,w)\} \end{bmatrix} \succ 0 \; \forall \; j \in \{1, \dots, n_u\}, \tag{66}$$

$$\mathrm{He}\left\{\begin{bmatrix} \hat{A}(z,w)X + B\hat{H}(z,w) & \hat{A}(z,w) + B\hat{K}(z,w)C & \Phi(z,w) + B\hat{K}(z,w)\Gamma & BD_m^\mathsf{T}\hat{E}(z,w) - BD^\mathsf{T}\hat{T} \\ \hat{F}(z,w) + \alpha I & Y\hat{A}(z,w) + \hat{G}(z,w)C & Y\Phi(z,w) + \hat{G}(z,w)\Gamma & \hat{W}(z,w) \\ L\Psi(z,w)X & L\Psi(z,w) & L\Omega(z,w) & 0 \\ \hat{R}(z,w) & \hat{\Xi}(z,w) & \hat{\Pi}(z,w) & -\hat{T} \end{bmatrix}\right\} \prec 0, \tag{67}$$

are verified $\forall \; (z,w) \in \mathcal{V}\{\mathcal{Z}^+\} \times \mathcal{V}\{\mathcal{W}^+\}$, where $\hat{A}(z,w) = A(z,w) + \alpha I$, matrices $M_1(z,w)$, $M_2(z,w)$, and $M_3(z,w)$ are

$$\begin{cases} M_1(z,w) \triangleq Q(z,w)X + D\hat{H}(z,w) - \hat{R}(z,w) \\ M_2(z,w) \triangleq Q(z,w) + D\hat{K}(z,w)C - \hat{\Xi}(z,w) \\ M_3(z,w) \triangleq \Upsilon(z,w) + D\hat{K}(z,w)\Gamma - \hat{\Pi}(z,w) \end{cases}, \tag{68}$$

matrices $\hat{F}(z,w)$, $\hat{G}(z,w)$, $\hat{H}(z,w)$, $\hat{K}(z,w)$, $\hat{E}(z,w)$, and $\hat{W}(z,w)$ are constructed in the form of

$$\begin{bmatrix} \hat{F}(z,w) & \hat{G}(z,w) \\ \hat{H}(z,w) & \hat{K}(z,w) \end{bmatrix} \triangleq \begin{bmatrix} \hat{\mathrm{F}}_0 & \hat{\mathrm{G}}_0 \\ \hat{\mathrm{H}}_0 & \hat{\mathrm{K}}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \hat{\mathrm{F}}_i & \hat{\mathrm{G}}_i \\ \hat{\mathrm{H}}_i & \hat{\mathrm{K}}_i \end{bmatrix} \lambda_i(z,w), \quad \begin{bmatrix} \hat{E}(z,w) \\ \hat{W}(z,w) \end{bmatrix} \triangleq \begin{bmatrix} \hat{\mathrm{E}}_0 \\ \hat{\mathrm{W}}_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \hat{\mathrm{E}}_i \\ \hat{\mathrm{W}}_i \end{bmatrix} \lambda_i(z,w), \tag{69}$$

and matrices $\hat{R}(z,w)$, $\hat{\Xi}(z,w)$, and $\hat{\Pi}(z,w)$ are defined such as

$$\begin{bmatrix} \hat{R}(z,w) & \hat{\Xi}(z,w) & \hat{\Pi}(z,w) \end{bmatrix} \triangleq \begin{bmatrix} \hat{\mathrm{R}}_0 & \hat{\Xi}_0 & \hat{\Pi}_0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} \hat{\mathrm{R}}_i & \hat{\Xi}_i & \hat{\Pi}_i \end{bmatrix} \nu_i(z,w), \tag{70}$$

for any linear function $\nu : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^m$, $m \in \mathbb{N}$. Suppose also that the matrices $N_1, N_2 \in \mathbb{R}^{n_z \times n_z}$ are any nonsingular solutions to $N_2 N_1^\mathsf{T} = I - XY$. Then, the trajectories of the closed-loop system (1), (3) with controller (10), (12), (26) parameterized as in (39) with

$$\begin{cases} \mathrm{F}_i = N_1^{-1}(\hat{\mathrm{F}}_i + YB\hat{\mathrm{K}}_i CX - \hat{\mathrm{G}}_i CX - YB\hat{\mathrm{H}}_i - Y\mathrm{A}_i X)N_2^{-\mathsf{T}} \\ \mathrm{G}_i = N_1^{-1}(\hat{\mathrm{G}}_i - YB\hat{\mathrm{K}}_i) \\ \mathrm{H}_i = (\hat{\mathrm{H}}_i - \hat{\mathrm{K}}_i CX)\, N_2^{-\mathsf{T}} \\ \mathrm{K}_i = \hat{\mathrm{K}}_i \end{cases} \quad \forall\, i \in \{0, \ldots, n\}, \tag{71}$$

$$\begin{cases} \mathrm{W}_0 = N_1^{-1}(\hat{\mathrm{W}}_0 - YBD_m^{\mathsf{T}}\hat{\mathrm{E}}_0)\hat{T}^{-1} + N_1^{-1}YBD^{\mathsf{T}} \\ \mathrm{W}_i = N_1^{-1}(\hat{\mathrm{W}}_i - YBD_m^{\mathsf{T}}\hat{\mathrm{E}}_i)\hat{T}^{-1} \quad \forall\, i \in \{1, \ldots, n\}\,, \\ \mathrm{E}_i = \hat{\mathrm{E}}_i\, \hat{T}^{-1} \quad \forall\, i \in \{0, \ldots, n\} \end{cases} \tag{72}$$

achieve output regulation with α-exponential convergence for any initial condition in

$$\mathcal{E} = \{(\mathbf{z}, w) \in \mathbb{R}^{n_a} \times \mathcal{W} \;:\; \mathbf{z}^{\mathsf{T}} P\, \mathbf{z} \le 1\}, \quad P \triangleq \begin{bmatrix} I & Y \\ 0 & N_1^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} X & I \\ N_2^{\mathsf{T}} & 0 \end{bmatrix}^{-1}. \tag{73}$$

Proof. Consider the candidate Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}^{\mathsf{T}} P\, \mathbf{z}\,. \tag{74}$$

If matrix P is symmetric and positive-definite, then $V(\mathbf{z}) > 0\ \forall\ \mathbf{z} \in \mathbb{R}^{n_a}$, $\mathbf{z} \neq 0$. The derivative of $V(\mathbf{z})$ along the trajectories of (42) is given by

$$\dot{V}(\mathbf{z}, w) = \mathrm{He}\{\mathbf{z}^{\mathsf{T}}\ \Delta_1(z, w)\ \zeta(\mathbf{z}, w)\}, \quad \Delta_1(z, w) \triangleq [P\boldsymbol{A}(z, w) \quad P\boldsymbol{\Phi}(z, w) \quad P\boldsymbol{J}(z, w)], \quad \zeta(\mathbf{z}, w) \triangleq \begin{bmatrix} \mathbf{z} \\ \varphi(z, w) \\ \psi(\mu) \end{bmatrix}. \tag{75}$$

Note that, for any matrix $L \in \mathbb{R}^{n_\varphi \times n_\varphi}$, the algebraic equality constraint from (42) can be reexpressed as

$$0 = \Delta_2(z, w)\ \zeta(\mathbf{z}, w), \quad \Delta_2(z, w) \triangleq L \begin{bmatrix} \boldsymbol{\Psi}(z, w) & \boldsymbol{\Omega}(z, w) & 0 \end{bmatrix}. \tag{76}$$

Recalling that

$$\mu = \boldsymbol{\theta}(\mathbf{z}, w) + c(w) = \boldsymbol{Q}(z, w)\ \mathbf{z} + \boldsymbol{\Upsilon}(z, w)\ \varphi(z, w) + c(w). \tag{77}$$

and considering $\boldsymbol{\vartheta} : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^{n_u}$ defined as

$$\boldsymbol{\vartheta}(\mathbf{z}, w) = \boldsymbol{R}(z, w)\ \mathbf{z} + \Pi(z, w)\ \varphi(z, w), \quad \boldsymbol{R}(z, w) \triangleq [R(z, w) \quad \Xi(z, w)], \tag{78}$$

where $R(z, w)$, $\Xi(z, w)$, and $\Pi(z, w)$ are affine matrix functions in (z, w) constructed such as

$$\begin{bmatrix} R(z, w) & \Xi(z, w) & \Pi(z, w) \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{R}_0 & \Xi_0 & \Pi_0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} \mathrm{R}_i & \Xi_i & \Pi_i \end{bmatrix} \nu_i(z, w), \tag{79}$$

for any linear function $\nu : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^m$, $m \in \mathbb{N}$. According to Lemma 3, if $(\mathbf{z}, w) \in \mathcal{S}$ with $\vartheta(\mathbf{z}, w)$ defined as in (78), it follows that (51) holds, that is

$$\psi^{\mathsf{T}}(\mu)\ \Delta_3(z, w)\ \zeta(\mathbf{z}, w) \ge 0, \quad \Delta_3(z, w) \triangleq T \begin{bmatrix} \boldsymbol{R}(z, w) & \Pi(z, w) & -I \end{bmatrix}, \tag{80}$$

with $T \in \mathbb{R}^{n_u \times n_u}$ being any diagonal and positive-definite matrix.

Now suppose that the following expression is true for some $\alpha > 0$, where $\boldsymbol{\mathcal{Z}}^+ \triangleq \mathcal{Z}^+ \times \mathbb{R}^{n_s}$:

$$\dot{V}(\mathbf{z}, w) + 2\alpha V(\mathbf{z}) + \mathrm{He}\{\varphi^{\mathsf{T}}(z, w)\ \Delta_2(z, w)\ \zeta(\mathbf{z}, w) + \psi^{\mathsf{T}}(\mu)\ \Delta_3(z, w)\ \zeta(\mathbf{z}, w)\} < 0 \quad \forall\ (\mathbf{z}, w) \in \boldsymbol{\mathcal{Z}}^+ \times \mathcal{W}^+,\ \mathbf{z} \neq 0, \tag{81}$$

which is equivalent to

$$\mathrm{He}\left\{\begin{bmatrix} P\boldsymbol{A}(z,w)+\alpha P & P\boldsymbol{\Phi}(z,w) & P\boldsymbol{J}(z,w) \\ L\boldsymbol{\Psi}(z,w) & L\boldsymbol{\Omega}(z,w) & 0 \\ T\boldsymbol{R}(z,w) & T\Pi(z,w) & -T \end{bmatrix}\right\} \prec 0 \;\; \forall\, (z,w) \in \mathcal{Z}^+ \times \mathcal{W}^+. \tag{82}$$

From this relation and taking into account (76) and (80), it follows that

$$\dot{V}(\mathbf{z},w) < -2\alpha V(\mathbf{z}) < 0 \;\forall\, (\mathbf{z},w) \in (\mathcal{Z}^+ \times \mathcal{W}^+) \cap \mathcal{S},\;\; \mathbf{z} \neq 0. \tag{83}$$

Our idea is therefore to determine a positively invariant region $\mathcal{E}$ such that for $\forall\, (\mathbf{z}(0), w(0)) \in \mathcal{E}$, the trajectories $(\mathbf{z}(t), w(t))$ never leave the sets $(\mathcal{Z}^+ \times \mathcal{W}^+)$ and $\mathcal{S}$. With this aim, consider $\mathcal{E} = \mathcal{Z} \times \mathcal{W}$ with $\mathcal{Z} \triangleq \{\mathbf{z} \in \mathbb{R}^{n_a} : \mathbf{z}^\mathsf{T} P\,\mathbf{z} \leq 1\}$ representing a level set of $V(\mathbf{z})$. Moreover, recall that, from Assumption 1, if $w(0) \in \mathcal{W}$ then $w(t) \in \mathcal{W} \subseteq \mathcal{W}^+$. Hence, two inclusion conditions need to be satisfied: (a) $\mathcal{Z} \subset \mathcal{Z}^+ = \mathcal{Z}^+ \times \mathbb{R}^{n_s}$ and (b) $\mathcal{E} \subset \mathcal{S}$.

Recalling that $\mathcal{Z}^+$ can be written in the form given in (34), the inclusion condition (a) is satisfied if and only if:[9]

$$\begin{bmatrix} 1 & q_k^\mathsf{T} \\ \star & P \end{bmatrix} \succ 0 \;\forall\, k \in \{1, 2, \ldots, n_k\}, \quad q_k \triangleq \begin{bmatrix} p_k \\ 0 \end{bmatrix}. \tag{84}$$

Now we derive a condition to ensure the inclusion (b). For this, suppose that the following relation is satisfied:

$$(\overline{u}_j - \overline{c}_j)^{-2}(\boldsymbol{\theta}_j(\mathbf{z},w) - \boldsymbol{\vartheta}_j(\mathbf{z},w))^\mathsf{T}(\boldsymbol{\theta}_j(\mathbf{z},w) - \boldsymbol{\vartheta}_j(\mathbf{z},w)) < \mathbf{z}^\mathsf{T} P\,\mathbf{z} \;\;\forall\, (\mathbf{z},w) \in \mathcal{Z}^+ \times \mathcal{W}^+,\; \forall\, j \in \{1,2,\ldots,n_u\}. \tag{85}$$

Thus, $\forall\, (\mathbf{z},w) \in \mathcal{Z} \times \mathcal{W}$, it is ensured from (50) that $(\mathbf{z},w) \in \mathcal{S}$, hence $\mathcal{E} = \mathcal{Z} \times \mathcal{W} \subset \mathcal{S}$. Taking into account the algebraic equality constraint in (76) and that $\boldsymbol{\theta}(\mathbf{z},w)$ and $\boldsymbol{\vartheta}(\mathbf{z},w)$ are as in (77) and (78), it follows that (85) is equivalent to

$$\mathbf{z}^\mathsf{T} P\,\mathbf{z} - \mathrm{He}\{\varphi^\mathsf{T}(z,w)\,\Delta_2(z,w)\,\zeta(\mathbf{z},w)\} - (\overline{u}_j - \overline{c}_j)^{-2}\,\zeta^\mathsf{T}(\mathbf{z},w)\,\Delta_{4_{[j]}}^\mathsf{T}(z,w)\,\Delta_{4[j]}(z,w)\,\zeta(\mathbf{z},w) > 0, \tag{86}$$

$\forall\, (\mathbf{z},w) \in \mathcal{Z}^+ \times \mathcal{W}^+,\; \forall\, j \in \{1,2,\ldots,n_u\}$, with

$$\Delta_4(z,w) \triangleq \begin{bmatrix} \boldsymbol{Q}(z,w) - \boldsymbol{R}(z,w) & \boldsymbol{\Upsilon}(z,w) - \Pi(z,w) & 0 \end{bmatrix}. \tag{87}$$

From Schür's complement, condition (85) is equivalent to

$$\begin{bmatrix} (\overline{u}_j - \overline{c}_j)^2 & \boldsymbol{Q}_{[j]}(z,w) - \boldsymbol{R}_{[j]}(z,w) & \boldsymbol{\Upsilon}_{[j]}(z,w) - \Pi_{[j]}(z,w) \\ \star & P & -\boldsymbol{\Psi}^\mathsf{T}(z,w)L^\mathsf{T} \\ \star & \star & -\mathcal{H}\{L\boldsymbol{\Omega}(z,w)\} \end{bmatrix} \succ 0 \;\;\forall\, (z,w) \in \mathcal{Z}^+ \times \mathcal{W}^+, \tag{88}$$

that is, (88) ensures that $\mathcal{E} \subset \mathcal{S}$.

Consequently, provided that $P \succ 0$, $T \succ 0$ and that the conditions (82), (84), and (88) hold, we can conclude that $\mathcal{E}$ is a positively invariant region. Moreover, it follows from (83) that:

$$V(\mathbf{z}(t)) \leq V(\mathbf{z}(0))\,\mathrm{e}^{-2\alpha t} \;\forall\, t \geq 0,\; \forall\, (\mathbf{z}(0), w(0)) \in \mathcal{E}. \tag{89}$$

Furthermore, since $\lambda_{\min}(P)\;\|\mathbf{z}\|^2 \leq V(\mathbf{z}) \leq \lambda_{\max}(P)|\mathbf{z}\|^2$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest and largest eigenvalues of P, we can also verify that relation (64) holds with

$$\beta = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} > 0. \tag{90}$$

Thus, the derived matrix inequality conditions $P > 0$, $T > 0$, (82), (84), and (88) imply that the trajectories $\mathbf{z}(t)$ of the system (42) exponentially converge to the origin with decay rate greater then α for every initial condition $(\mathbf{z}(0), w(0)) \in \mathcal{E} = \mathcal{Z} \times \mathcal{W}$. From Lemma 2, it follows that the trajectories of the closed-loop system (1), (3) with controller (10), (12), (26) achieves output regulation in the region

$$\mathcal{D} = \left\{ (x, \xi_m, \xi_s, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_s} \times \mathcal{W} : \begin{bmatrix} x - \pi(w) \\ \xi_m - \sigma_m(w) \\ \xi_s \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} x - \pi(w) \\ \xi_m - \sigma_m(w) \\ \xi_s \end{bmatrix} \leq 1 \right\}. \tag{91}$$

Next, following the parametrizations proposed in Reference 31, we show that considering a full order stabilizing controller (i.e., $n_s = n_z$), conditions $P > 0$, $T > 0$, (82), (84), and (88) are equivalent to the ones in (65), (66), and (67). For this, consider the following definitions:

$$\begin{bmatrix} Y & N_1 \\ N_1^{\mathsf{T}} & \cdot \end{bmatrix} \triangleq P, \quad \begin{bmatrix} X & N_2 \\ N_2^{\mathsf{T}} & \cdot \end{bmatrix} \triangleq P^{-1}, \quad Z_1 \triangleq \begin{bmatrix} I & Y \\ 0 & N_1^{\mathsf{T}} \end{bmatrix}, \quad Z_2 \triangleq \begin{bmatrix} X & I \\ N_2^{\mathsf{T}} & 0 \end{bmatrix}, \tag{92}$$

noting that $Z_1 = PZ_2$, $P = Z_1 Z_2^{-1}$ and that the matrix pair N_1, N_2 is any solution to $N_2 N_1^{\mathsf{T}} = I - XY$, so as to satisfy the identity $PP^{-1} = I$. By post- and pre-multiplying the matrix inequalities $P > 0$, $T > 0$, (82), (84), and (88), respectively, by Z_2, T^{-1}, $\text{diag}\{Z_2, I, T^{-1}\}$, $\text{diag}\{1, Z_2\}$, $\text{diag}\{1, Z_2, I\}$, and their transposes, one obtains therefore the conditions in (65), (66), and (67) considering the following change of variables:

$$\begin{cases} \hat{F}(z, w) \triangleq YA(z, w)X + YBK(z, w)CX + N_1 G(z, w)CX + YBH(z, w)N_2^{\mathsf{T}} + N_1 F(z, w)N_2^{\mathsf{T}} \\ \hat{G}(z, w) \triangleq YBK(z, w) + N_1 G(z, w) \\ \hat{H}(z, w) \triangleq K(z, w)CX + H(z, w)N_2^{\mathsf{T}} \\ \hat{K}(z, w) \triangleq K(z, w) \\ \hat{E}(z, w) \triangleq E(z, w)T^{-1} \\ \hat{W}(z, w) \triangleq YBD_m^{\mathsf{T}}E(z, w)T^{-1} - YBD^{\mathsf{T}}T^{-1} + N_1 W(z, w)T^{-1} \\ \hat{R}(z, w) \triangleq R(z, w)X + \Xi(z, w)N_2^{\mathsf{T}} \\ \hat{\Xi}(z, w) \triangleq R(z, w) \\ \hat{\Pi}(z, w) \triangleq \Pi(z, w) \\ \hat{T} \triangleq T^{-1} \end{cases}. \tag{93}$$

From the second condition in (65), it follows that X and Y are nonsingular matrices. Then, it is always possible to find nonsingular matrices N_1 and N_2 verifying $N_2 N_1^{\mathsf{T}} = I - XY$. Hence, in this case, the original controller matrices can always be recovered as in (71) and (72). Moreover, as X, Y, N_1, and N_2 are nonsingular, matrix P of the Lyapunov function can always be recovered as in (73), which is consequently ensured to be nonsingular and positive-definite.

Recalling that all system matrices $A(z, w)$, $\Phi(z, w)$, …, $\Omega(z, w)$ are affine functions in (z, w), in order to ensure that conditions (66) and (67) hold $\forall\ (z, w) \in \mathcal{Z}^+ \times \mathcal{W}^+$, it is necessary and sufficient to verify these inequalities just on the vertices of regions $\mathcal{Z}^+$ and $\mathcal{W}^+$, that is, $\forall\ (z, w) \in \mathcal{V}\{\mathcal{Z}^+\} \times \mathcal{V}\{\mathcal{W}^+\}$, which concludes the proof. ■

Based on Theorem 1, a problem of interest regards the synthesis of the stabilizing controller and anti-windup gains leading to a maximized domain of attraction estimate $\mathcal{E}$ in the direction of the plant and internal model regulation error states z, considering that the initial condition of the stabilizing controller states is set as zero, that is, $\xi_s(0) = 0$. In this case, it follows from (92) that $\mathbf{z}^{\mathsf{T}}(0)P\,\mathbf{z}(0) = z^{\mathsf{T}}(0)Yz(0)$ and, consequently, the target domain of attraction estimate to be maximized is simplified from (73) to:

$$\mathcal{E}_z = \{(z, w) \in \mathbb{R}^{n_z} \times \mathcal{W} : z^{\mathsf{T}} Y z \leq 1\}. \tag{94}$$

This task can be accomplished, for instance, by minimizing the trace of matrix Y leading to the following optimization problem:

$$\min_{X,Y,L,\hat{T},\hat{R}_0,\ldots,\hat{\Pi}_m,\hat{F}_0,\ldots,\hat{W}_n} \operatorname{tr}(Y) \text{ s.t. } \{(65),\ (66),\ (67)\}\ \forall\ (z,w) \in \mathcal{V}\{\mathcal{Z}^+\} \times \mathcal{V}\{\mathcal{W}^+\}. \tag{95}$$

Note that (95) is not convex due to the bilinearities involving the pair of decision variables L and X in (66) and (67). However, if either of these matrices is a priori fixed, then (95) becomes a standard semidefinite optimization problem, that is, a linear criterion subject to LMI constraints. This idea can be employed in order to iteratively find a locally optimal solution, similar to the so-called D-K iteration method.[32] On the other hand, considering a particular case, it is possible to reformulate the provided stability conditions stated in Theorem 1 in order to readily obtain a convex numerical optimization with LMI constraints.

Assumption 4. Suppose the following additional conditions hold with respect to the DAR introduced in (35) and (36):

- There exists a function $\varphi(y)$, $\varphi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_\varphi}$, such that

$$\varphi(g_z(z,w) + d(w)) = \varphi(z,w). \tag{96}$$

- Matrix $\Phi(z,w)$ can be expressed as

$$\Phi(z,w) = \Phi_0 + \sum_{i=1}^{n} \Phi_i\ \lambda_i(z,w). \tag{97}$$

  with constant matrices $\Phi_0, \ldots, \Phi_n \in \mathbb{R}^{n_z \times n_\varphi}$.

Assumption 4 means that all rational nonlinearities contained in functions $f_z(z,w,v)$, $\theta_z(z,w,v)$, and $\delta_z(z,w)$ can be constructed by a proper mapping of the measurement vector y. A similar assumption has been considered in Reference 33 to cast the output feedback stabilization as a convex optimization problem.

Provided that Assumption 4 is true, the original stabilizing stage (26) can be modified to include the nonlinear function $\varphi(y)$ according to the following structure:

$$\begin{cases} \dot{\xi}_s = \mathrm{F}(y)\ \xi_s + \mathrm{G}(y)\ \varepsilon + \Lambda(y)\ \varphi(y) + \mathrm{W}(y)\ \psi(\mu) \\ v = \mathrm{H}(y)\ \xi_s + \mathrm{K}(y)\ \varepsilon + \Theta(y)\ \varphi(y) \end{cases}, \tag{98}$$

where terms $\Lambda : \mathbb{R}^{n_y} \to \mathbb{R}^{n_s \times n_\varphi}$ and $\Theta : \mathbb{R}^{n_y} \to \mathbb{R}^{n_v \times n_\varphi}$ are additional free design matrix functions in the following affine form:

$$\begin{bmatrix} \Lambda(y) \\ \Theta(y) \end{bmatrix} \triangleq \begin{bmatrix} \Lambda_0 \\ \Theta_0 \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} \Lambda_i \\ \Theta_i \end{bmatrix} \lambda_i(y)\ . \tag{99}$$

Now considering (98), the augmented system matrices $\mathbf{\Phi}(z,w)$ and $\mathbf{\Upsilon}(z,w)$—previously defined as in (43)—are modified as

$$\mathbf{\Phi}(z,w) = \begin{bmatrix} \Phi(z,w) + BK(z,w)\Gamma + B\Theta(z,w) \\ G(z,w)\Gamma + \Lambda(z,w) \end{bmatrix}, \quad \mathbf{\Upsilon}(z,w) = \Upsilon(z,w) + DK(z,w)\Gamma + D\Theta(z,w), \tag{100}$$

where $\Lambda(z,w)$ and $\Theta(z,w)$ denote the controller matrices in (99) expressed in terms of (z,w), similarly to (40).

By using the newer definitions in (100), the following corollary can be derived, which is an adaptation of Theorem 1 with the additional feedback parameters involving the function $\varphi(y)$.

Corollary 1. *Consider that Assumption 4 is verified and assume $n_s = n_z$. Suppose that there exist symmetric matrices $X, Y \in \mathbb{R}^{n_z \times n_z}$, a diagonal matrix $\hat{T} \in \mathbb{R}^{n_u \times n_u}$ and matrices $\hat{L} \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $\hat{\mathrm{R}}_0, \ldots, \hat{\mathrm{R}}_m \in \mathbb{R}^{n_u \times n_z}$, $\hat{\Xi}_0, \ldots, \hat{\Xi}_m \in \mathbb{R}^{n_u \times n_s}$, $\hat{\Pi}_0, \ldots, \hat{\Pi}_m \in \mathbb{R}^{n_u \times n_\varphi}$, $\hat{\mathrm{F}}_0, \ldots, \hat{\mathrm{F}}_n \in \mathbb{R}^{n_z \times n_z}$, $\hat{\mathrm{G}}_0, \ldots, \hat{\mathrm{G}}_n \in \mathbb{R}^{n_z \times n_\varepsilon}$, $\hat{\mathrm{H}}_0, \ldots, \hat{\mathrm{H}}_n \in \mathbb{R}^{n_v \times n_z}$, $\hat{\mathrm{K}}_0, \ldots, \hat{\mathrm{K}}_n \in \mathbb{R}^{n_v \times n_\varepsilon}$, $\hat{\mathrm{E}}_0, \ldots, \hat{\mathrm{E}}_n \in \mathbb{R}^{n_m \times n_u}$ and $\hat{\mathrm{W}}_0, \ldots, \hat{\mathrm{W}}_n \in \mathbb{R}^{n_z \times n_u}$, $\hat{\Lambda}_0, \ldots, \hat{\Lambda}_n \in \mathbb{R}^{n_z \times n_\varphi}$ and $\hat{\Theta}_0, \ldots, \hat{\Theta}_n \in \mathbb{R}^{n_v \times n_\varphi}$ such that the matrix inequalities: (65),*

$$\begin{bmatrix} (\overline{u}_j - \overline{c}_j)^2 & M_{4[j]}(z,w) & M_{5[j]}(z,w) & M_{6[j]}(z,w) \\ \star & X & I & -X\Psi^{\mathsf{T}}(z,w) \\ \star & \star & Y & -\Psi^{\mathsf{T}}(z,w) \\ \star & \star & \star & -\mathrm{He}\{\Omega(z,w)\hat{L}\} \end{bmatrix} \succ 0, \tag{101}$$

$$\mathrm{He}\left\{\begin{bmatrix} \hat{A}(z,w)X + B\hat{H}(z,w) & \hat{A}(z,w) + B\hat{K}(z,w)C & \Phi(z,w)\hat{L} + B\hat{\Theta}(z,w) & BD_m^{\mathsf{T}}\hat{E}(z,w) - BD^{\mathsf{T}}\hat{T} \\ \hat{F}(z,w) + \alpha I & Y\hat{A}(z,w) + \hat{G}(z,w)C & \hat{\Lambda}(z,w) & \hat{W}(z,w) \\ \Psi(z,w)X & \Psi(z,w) & \Omega(z,w)\hat{L} & 0 \\ \hat{R}(z,w) & \hat{\Xi}(z,w) & \hat{\Pi}(z,w) & -\hat{T} \end{bmatrix}\right\} \prec 0, \tag{102}$$

are verified $\forall\ (z,w) \in \mathcal{V}\{\mathcal{Z}^+\} \times \mathcal{V}\{\mathcal{W}^+\}$, *where* $\hat{A}(z,w) = A(z,w) + \alpha I$, *matrices* $M_4(z,w)$, $M_5(z,w)$, *and* $M_6(z,w)$ *are*

$$\begin{cases} M_4(z,w) \triangleq Q(z,w)X + D\hat{H}(z,w) - \hat{R}(z,w) \\ M_5(z,w) \triangleq Q(z,w) + D\hat{K}(z,w)C - \hat{\Xi}(z,w)\ , \\ M_6(z,w) \triangleq \Upsilon(z,w)\hat{L} + D\hat{\Theta}(z,w) - \hat{\Pi}(z,w) \end{cases} \tag{103}$$

matrices $\hat{F}(z,w)$, $\hat{G}(z,w)$, $\hat{H}(z,w)$, $\hat{K}(z,w)$, $\hat{E}(z,w)$, $\hat{W}(z,w)$, $\hat{\Lambda}(z,w)$, *and* $\hat{\Theta}(z,w)$ *are constructed in the form of (69) and matrix functions* $\hat{R}(z,w)$, $\hat{\Xi}(z,w)$, *and* $\hat{\Pi}(z,w)$ *are defined such as in (70) for any linear function* $\nu : \mathcal{Z}^+ \times \mathcal{W}^+ \to \mathbb{R}^m$, $m \in \mathbb{N}$. *Suppose also that matrices* $N_1, N_2 \in \mathbb{R}^{n_z \times n_z}$ *are any nonsingular solutions to* $N_2 N_1^{\mathsf{T}} = I - XY$. *Then, the trajectories of the closed-loop system (1), (3) with controller (10), (12), (98) parameterized as in (39) and (99) with matrices given in (71), (72) and*

$$\begin{cases} \Lambda_i = N_1^{-1}(\hat{\Lambda}_i - YB\hat{\Theta}_i)\hat{L}^{-1} + N_1^{-1}(YB\hat{\mathrm{K}}_i - \hat{\mathrm{G}}_i)\Gamma - N_1^{-1}Y\Phi_i \\ \Theta_i = \hat{\Theta}_i\hat{L}^{-1} - \hat{\mathrm{K}}_i\Gamma \end{cases} \quad \forall\ i \in \{0, \ldots, n\}\ , \tag{104}$$

achieve output regulation with α*-exponential convergence for any initial condition in (73).*

Proof. Follow the same steps presented in the proof of Theorem 1, however, post- and pre-multiply (82) and (88), respectively, by $\mathrm{diag}\{Z_2, L^{-\mathsf{T}}, T^{-1}\}$, $\mathrm{diag}\{1, Z_2, L^{-\mathsf{T}}\}$ and their transposes, noting that $\mathbf{\Phi}(z,w)$ and $\mathbf{\Upsilon}(z,w)$ are now given as in (100). Moreover, consider the additional change of variables

$$\begin{cases} \hat{\Lambda}(z,w) \triangleq (Y\Phi(z,w) + YB\Theta(z,w) + YBK(z,w)\Gamma + N_1\Lambda(z,w) + N_1 G(z,w)\Gamma)L^{-\mathsf{T}} \\ \hat{\Theta}(z,w) \triangleq (\Theta(z,w) + K(z,w)\Gamma)L^{-\mathsf{T}} \\ \hat{\Pi}(z,w) \triangleq \Pi(z,w)L^{-\mathsf{T}} \\ \hat{L} \triangleq L^{-\mathsf{T}} \end{cases} . \tag{105}$$

This alternative procedure yields the relations (101) and (102) instead of (66) and (67). It also follows that the additional stabilizing controller parameters can always be recovered as in (104), which concludes the proof. ∎

In order to maximize the domain of attraction estimate (94), based on Corollary 1, one can synthesize the stabilizing and anti-windup controller parameters from the following convex optimization problem:

$$\min_{X,Y,\hat{L},\hat{T},\hat{\mathrm{R}}_0,\ldots,\hat{\Pi}_m,\hat{\mathrm{F}}_0,\ldots,\hat{\Theta}_n} \mathrm{tr}(Y)\ \text{ s.t. }\ \{(65),\ (101),\ (102)\}\ \forall\ (z,w) \in \mathcal{V}\{\mathcal{Z}^+\} \times \mathcal{V}\{\mathcal{W}^+\}. \tag{106}$$

To illustrate our main results, in the sequence we consider the numerical case study previously dealt in Examples 1, 2 and 3. In here, the solution of the output regulation problem will finally be completed with the co-design of the stabilizing controller and anti-windup parameters from the proposed optimization problems.

Remark 2. By increasing the exponential decay rate α, one should expect to synthesize a more aggressive controller with larger feedback gains, making the saturation effect more pronounced. One should also expect to obtain a smaller domain of attraction estimate, which will inevitably restrict the range of admissible initial conditions.[34]

Example 4. Consider the rational nonlinear plant and the exosystem presented in (20) with numerical parameters as in Table 1, with zero-error steady-state mappings as shown in (22) and with an internal model stage controller as defined in (23). Suppose also that the stabilization problem is arranged as explained in Example 2, where the regulation error system was described using the rational nonlinear function $\varphi(z)$ from (46) and using the DAR matrices presented in (47) and (48). Consider also the preliminary remarks mentioned in Example 3, where the upper-bound $\bar{c} = 1.1388$ was determined considering the system parameters from Table 1. Note that parameter $\bar{u} = 4$ from Table 1 is the one directly related to the control saturation effect, since it denotes the maximum magnitude value of the control input signal $u(t)$.

Recall that the trajectories $w(t)$ of the Lorenz exosystem are contained in a spherical positively invariant set with radius r and center point w_c as declared in (21). Since the DAR matrices depend only on w_1, recall also that the set $\mathcal{W}^+$ does not need to restrict the w_2 and w_3 dimensions. For instance, one may consider $\mathcal{W}^+ = [-r,\ r] \times \mathbb{R}^2 \supset \mathcal{W}$. Likewise, since the z-dependence of $\Omega(z)$ is just involving z_1, the set $\mathcal{Z}^+$ only need to restrict the first dimension of the z-state-space. In this case, one can define $\mathcal{Z}^+ = [-\bar{z}_1,\ \bar{z}_1] \times \mathbb{R}^4$, for some constant $\bar{z}_1 > 0$ which denotes the maximum admissible value of $|z_1(t)| = |e(t)|\ \forall\, t \geq 0$. At last, the equivalent form of $\mathcal{Z}^+$ as (34) is obtained with $n_k = 1$ and $p_1 = [\bar{z}_1^{-1}\ \ 0\ \ 0\ \ 0\ \ 0]^\mathsf{T}$. According to Theorem 1, there is an additional free design choice with respect to function $\nu(z, w)$. Because the considered sets $\mathcal{Z}^+$ and $\mathcal{W}^+$ impose restrictions on the dimensions z_1 and w_1 solely, the function $\nu(z, w)$ may naturally be defined as $\nu(z, w) = [z_1\ \ w_1]^\mathsf{T}$.

The design specification for the maximum admissible output error amplitude was arbitrarily set as $\bar{z}_1 = 10^4$, whereas the minimum exponential decay rate was declared as $\alpha = 5 \cdot 10^{-2}$. Considering the setup above, optimization (95) was evaluated in order to synthesize the stabilizing and anti-windup gains, considering the general stabilizing stage (13). So as to deal with the BMI constraint in (95), matrices L and X were iteratively alternated between a priori fixed and decision variables. By initializing $L = -10^{-4} I$, it took 10 iterations in order to achieve a proximate locally optimal solution with the objective value $\mathrm{tr}(Y) = 1.0435$.

A noticeable property of this numerical example is that the rational function $\varphi(z)$, as shown in (46), may be remapped with respect to the output measurement y, that is,

$$\varphi(z) = \begin{bmatrix} z_1^2\ (1 + z_1^2)^{-1} \\ z_1\ (1 + z_1^2)^{-1} \end{bmatrix}, \quad y_1 = z_1 \quad \Rightarrow \quad \varphi(y) = \begin{bmatrix} y_1^2\ (1 + y_1^2)^{-1} \\ y_1\ (1 + y_1^2)^{-1} \end{bmatrix}. \tag{107}$$

Assumption 4 is therefore verified and it is possible to implement the stabilizing stage as stated in (98) in order to possibly achieve a larger domain of attraction estimate. Beyond this advantage, the controller synthesis can be addressed directly by a convex optimization with LMI constraints, as indicated by (106). The optimal controller design in this case was obtained with the objective value $\mathrm{tr}(Y) = 0.0161$, substantially smaller when compared with the one obtained considering the solution of (95).

Figures 2 and 3 present numerical simulations of the closed-loop system (20) considering the internal model controller (23) and the stabilizing stage (98) with parameters designed according to the proposed methodology. For these numerical analyses, the exosystem initial state was considered as the randomly picked point $w(0) = [3.5910\ \ \ 6.7150\ \ \ 9.4426]^\mathsf{T}$ inside the invariant sphere $\mathcal{W}$, while the controller initial states were set as $\xi_m(0) = 0$ and $\xi_s(0) = 0$, conditions which will be henceforth referred as default. In order to visualize the effect due to control input saturation, two different plant initial states $x(0) = [5\ \ \ 2.79 \cdot 10^3]^\mathsf{T}$ and $x(0) = [-5\ \ \ -2.79 \cdot 10^3]^\mathsf{T}$ were evaluated, which leads $(\mathbf{z}(0), w(0))$ marginally close to the border of the estimated domain of attraction $\mathcal{E}$ and respectively, correspond to the black and gray trajectories in Figures 2 and 3. This attained domain of attraction estimate is subsequently depicted (with respect to the x-state-space) by the black contour in Figure 4, where the simulated initial states are represented by the bold dots.

Figure 2 shows on top the system output error signals $e(t)$, where asymptotic convergence to the origin is verified as expected. The bottom plots of the same figure also detail the control input signals $u(t)$ produced by the designed controller, where one can clearly observe the saturation effect during the transient phase. After the initial saturated period, the input signals smoothly converges to the nonvanishing excitation $c(w(t))$ required to achieve output regulation. The phase portraits in Figure 3 illustrate the internal model trajectories $\xi_m(t)$ achieved during the simulations. Since the internal model zero-error steady-state is described by $\sigma_m(w(t)) = a_2 a_3^{-1}\ w(t)$, the trajectories $\xi_m(t)$ are expected to asymptotically approach the peculiar chaotic attractor of the Lorenz exosystem scaled by the factor $a_2 a_3^{-1}$, what is being illustrated in the right plot of Figure 3.

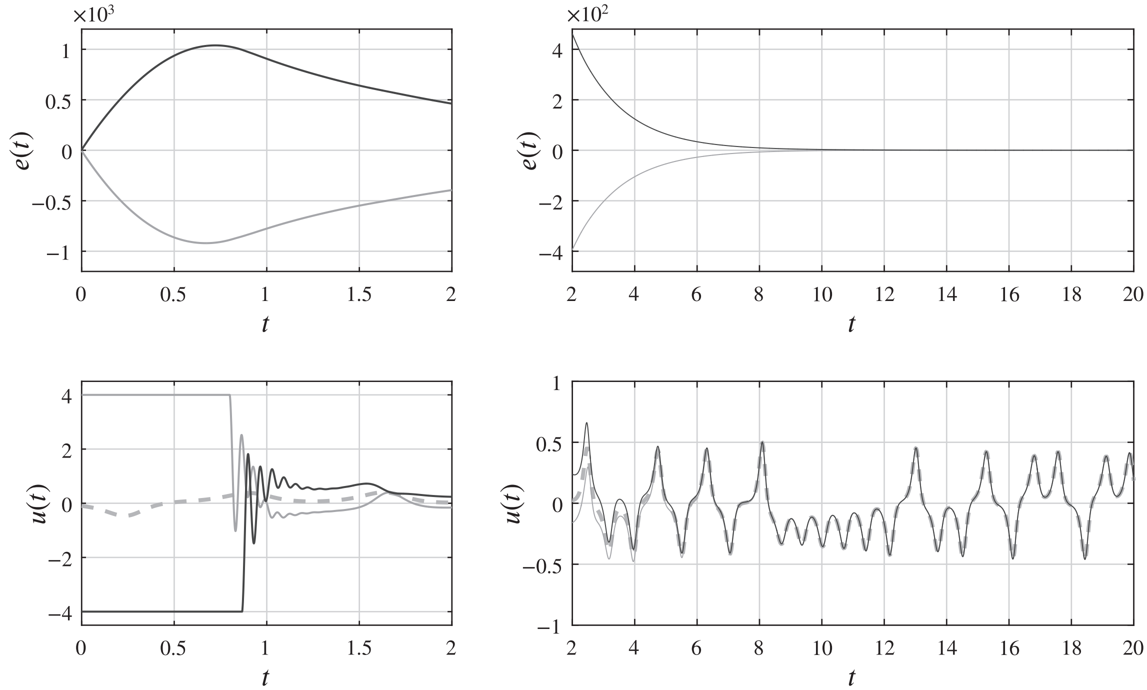

FIGURE 2 Top plots show the output regulation error signals $e(t)$, while bottom plots depict the control input signals $u(t)$ compared with the zero-error steady-state waveform $c(w(t))$ (dashed line). Left and right plots depict the transient and the steady state responses, respectively

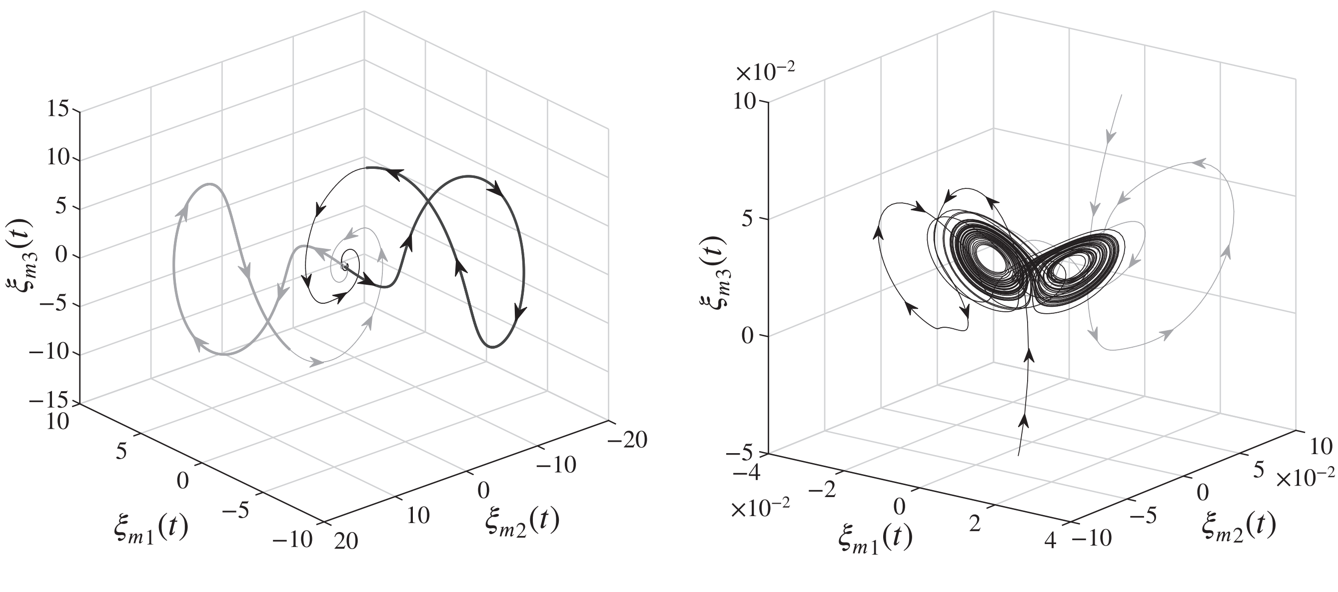

FIGURE 3 Left plot presents the internal model states trajectories $\xi_m(t)$ during the transient phase, where thick segments denote the initial control saturation period. Right plot in turn depicts the steady-state settling of the trajectories $\xi_m(t)$

The set of plant initial states $x(0)$ with guaranteed output regulation and exponential convergence is represented by the black contour in Figure 4, which corresponds to the region (73), where the initial state for the exosystem and the controller were considered fixed at the default conditions previously mentioned. For comparison purposes, the cross and circle grid depicts the estimated domain of attraction in case the proposed anti-windup compensation is disabled, which in this case was evaluated by numerical simulations. One may clearly observe the gray crossings inside the black contour, cases where the anti-windup action was verified to be necessary in order to maintain the closed-loop stability.

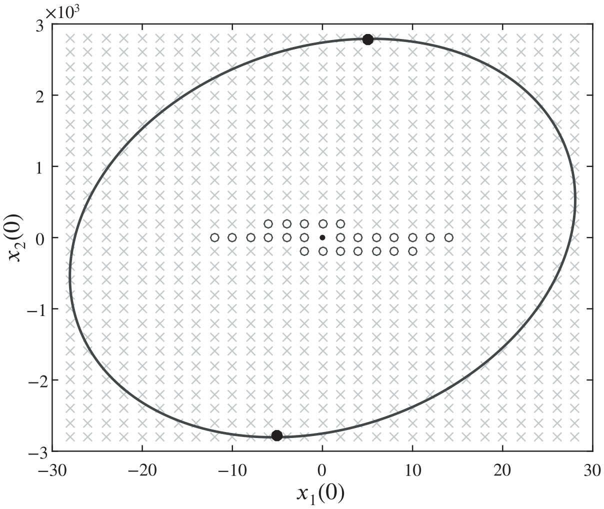

FIGURE 4 Black contour denotes the projection of the domain of attraction estimate $\mathcal{E}$ into the x-state-space, where the designed controller ensures output regulation and exponential performance. Bold dots denote the initial conditions $x(0)$ considered for the numerical simulations presented by Figures 2 and 3. Crosses represent initial conditions for which the closed-loop system became unstable if the anti-windup compensation is deactivated. The small dot is an initial condition which did not produce input saturation $\forall\, t \geq 0$, whereas circles denote cases which produced input saturation and the closed-loop-system without anti-windup action still achieved output regulation

4 | CONCLUSION

This article dealt with the output regulation of rational nonlinear systems subject to control input saturation. The control architecture is composed by an internal model stage in series with an output feedback stabilizing controller, where anti-windup compensation is considered for both of these stages. A set of matrix inequalities conditions were derived to ensure the asymptotic output regulation and the exponential convergence of the closed-loop system trajectories. These conditions in turn led to a systematic methodology for the simultaneous design of the stabilizing controller gains and the parameters of anti-windup loops acting on the internal model and the stabilizing controller.

The nonlinear output regulation control design problem, in the presence of input saturation, has been still open in the literature. The work presented in this article provided a significant advancement in this research topic, since beyond addressing saturation and rational nonlinearities, we also derived a complete co-design framework with built-in anti-windup compensation into all the controller states. Moreover, it is worth highlighting that our work does not require a strict triangular form representation or any other structural restriction rather than the rationality of the regulation error dynamics, since the proposed stabilization methodology is based on the solution of numerical optimization problems.

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CONFLICT OF INTEREST
The authors declare no potential conflict of interest.

ORCID
Rafael S. Castro https://orcid.org/0000-0002-8001-5194
Jeferson V. Flores https://orcid.org/0000-0003-2188-4661
Aurélio T. Salton https://orcid.org/0000-0002-5814-0853
João M Gomes da Silva Jr. https://orcid.org/0000-0002-6275-2895

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